Variance Analysis for Monte Carlo Integration

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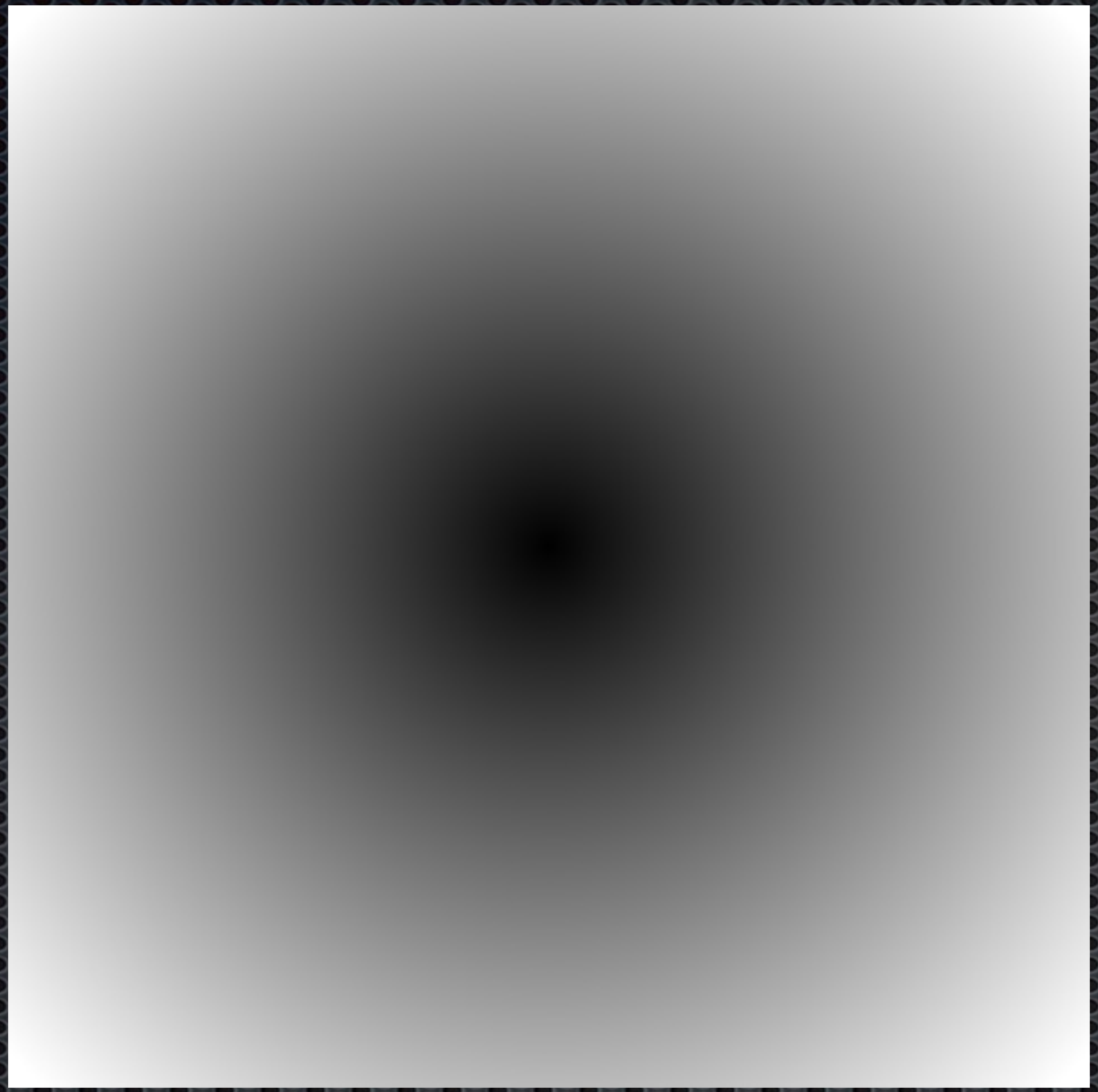
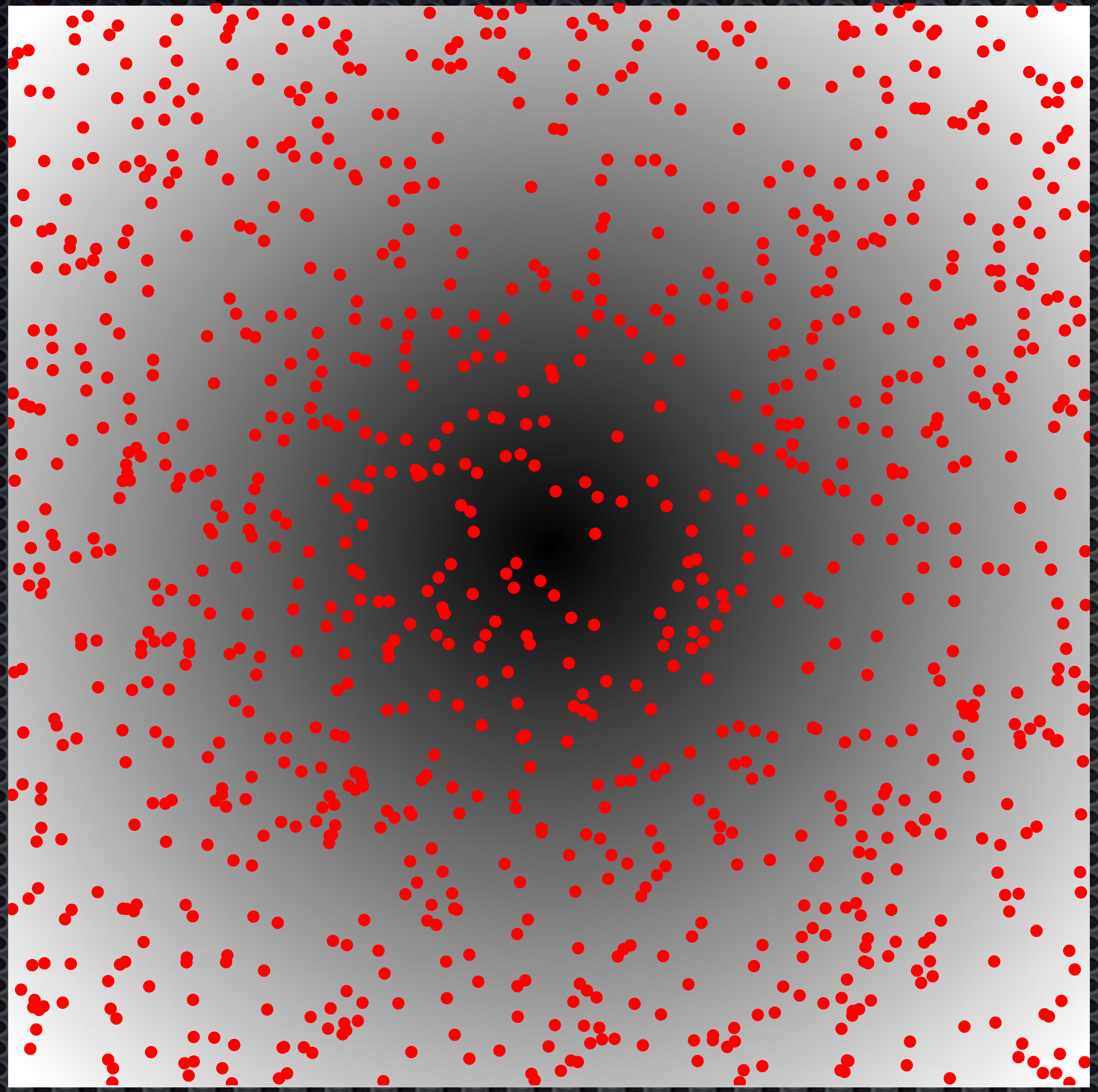
Monte Carlo Integration



$f(x)$

$$\int_{[0,1]^2} f(x)dx$$

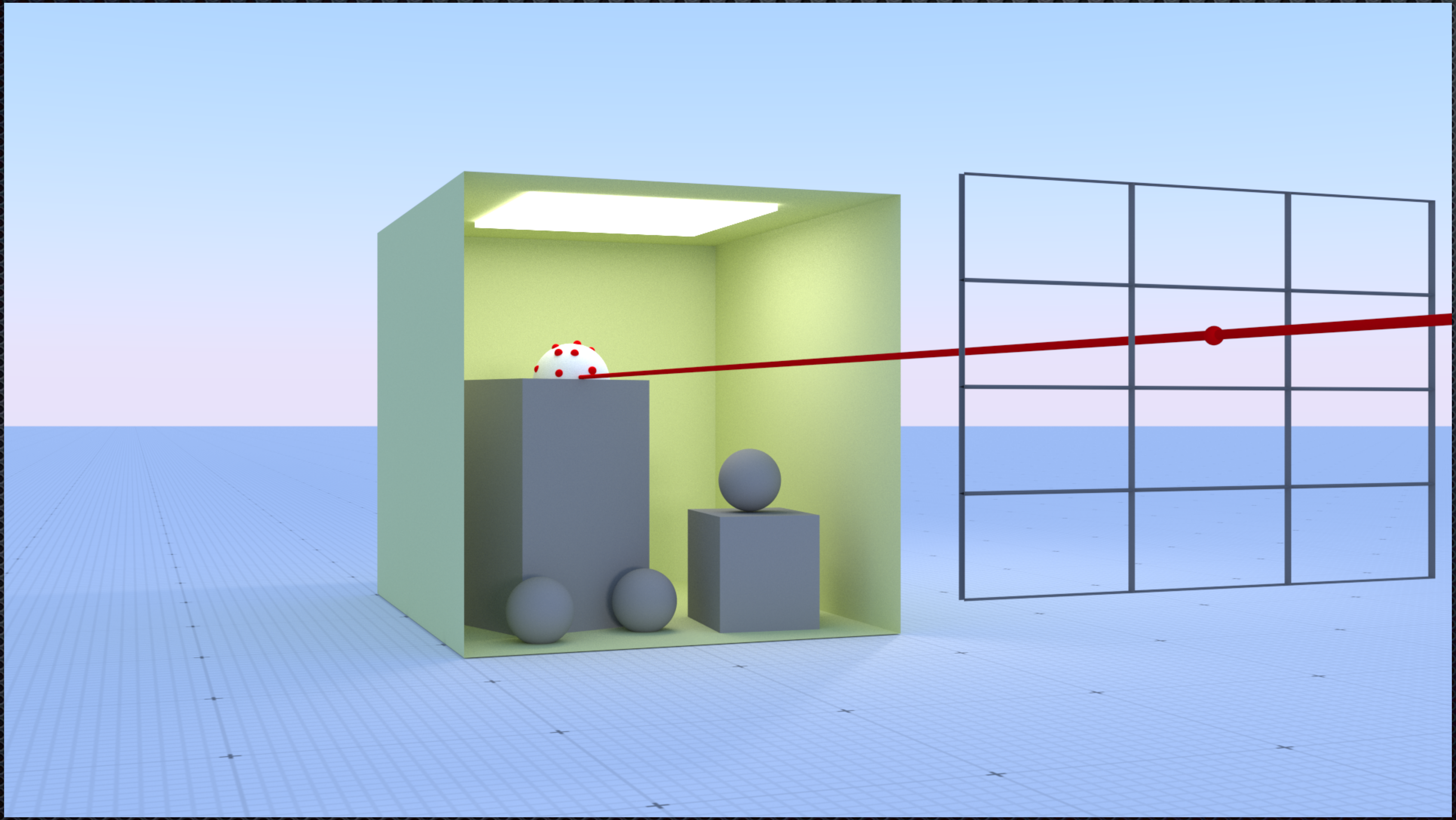
Monte Carlo Integration



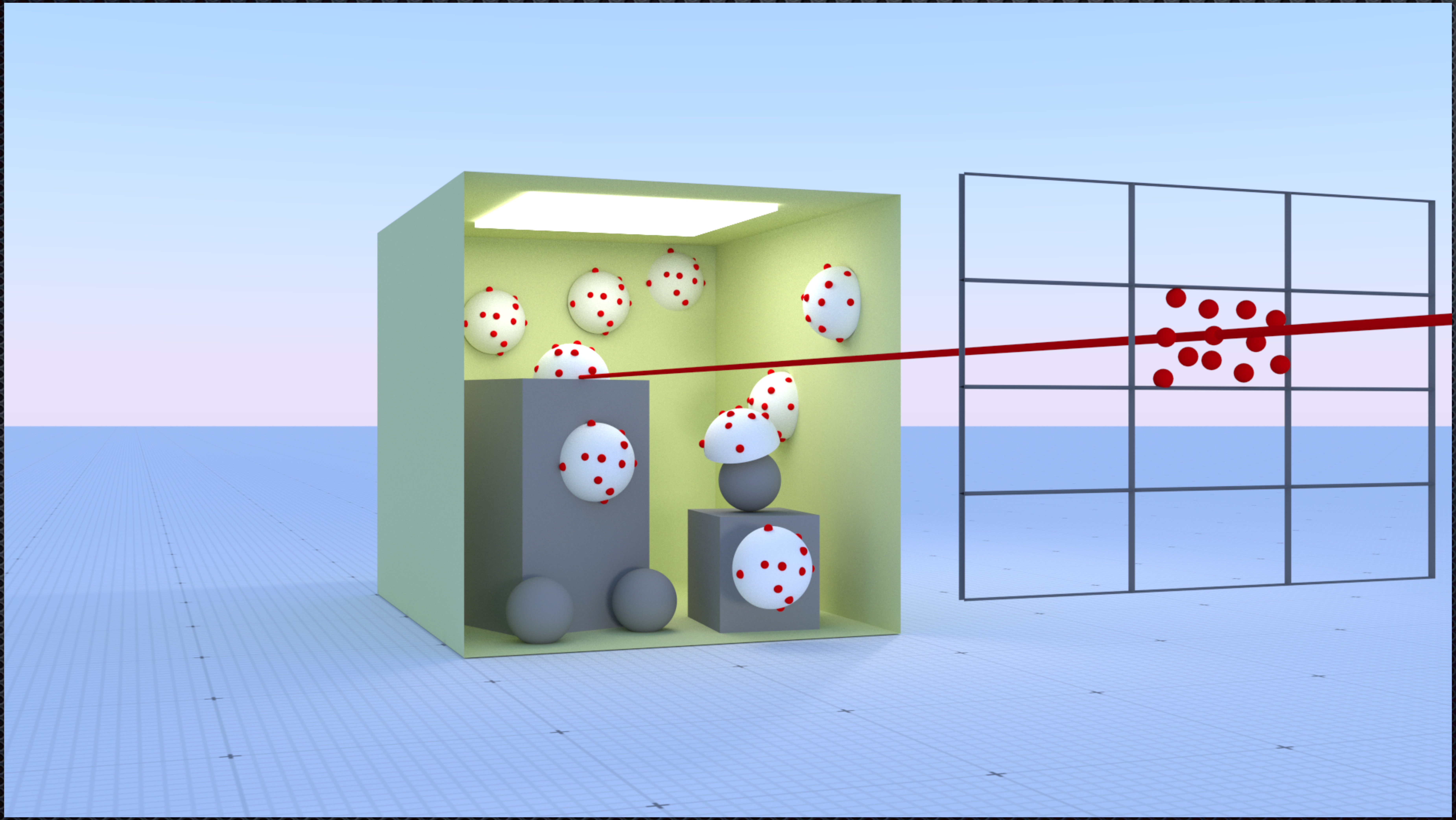
$f(x)$

$$\int_{[0,1]^2} f(x)dx \approx \frac{1}{N}\sum_{i=1}^{N} f(x_i)$$

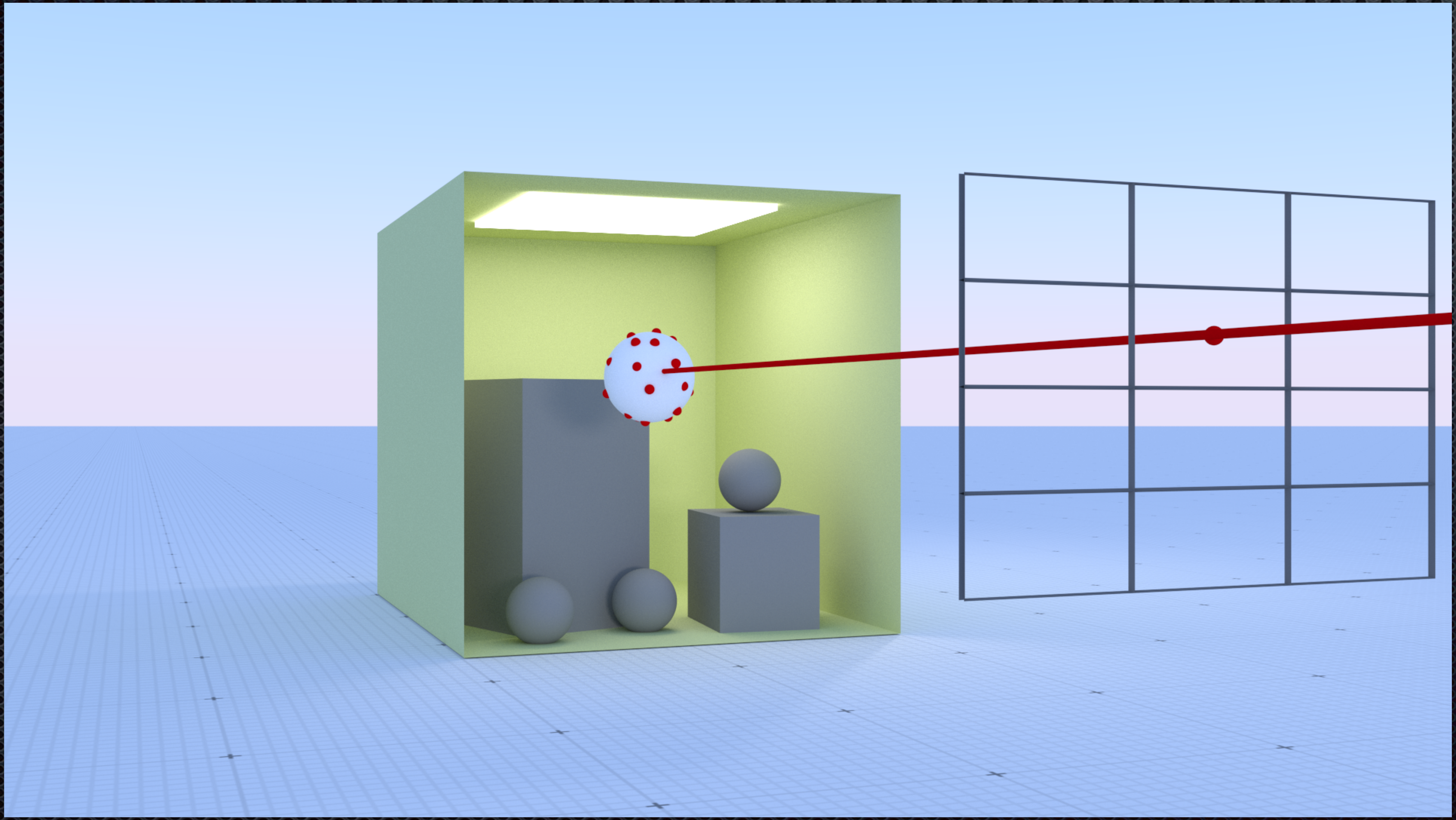
Light Simulation: on Surface



Light Simulation: on Surface

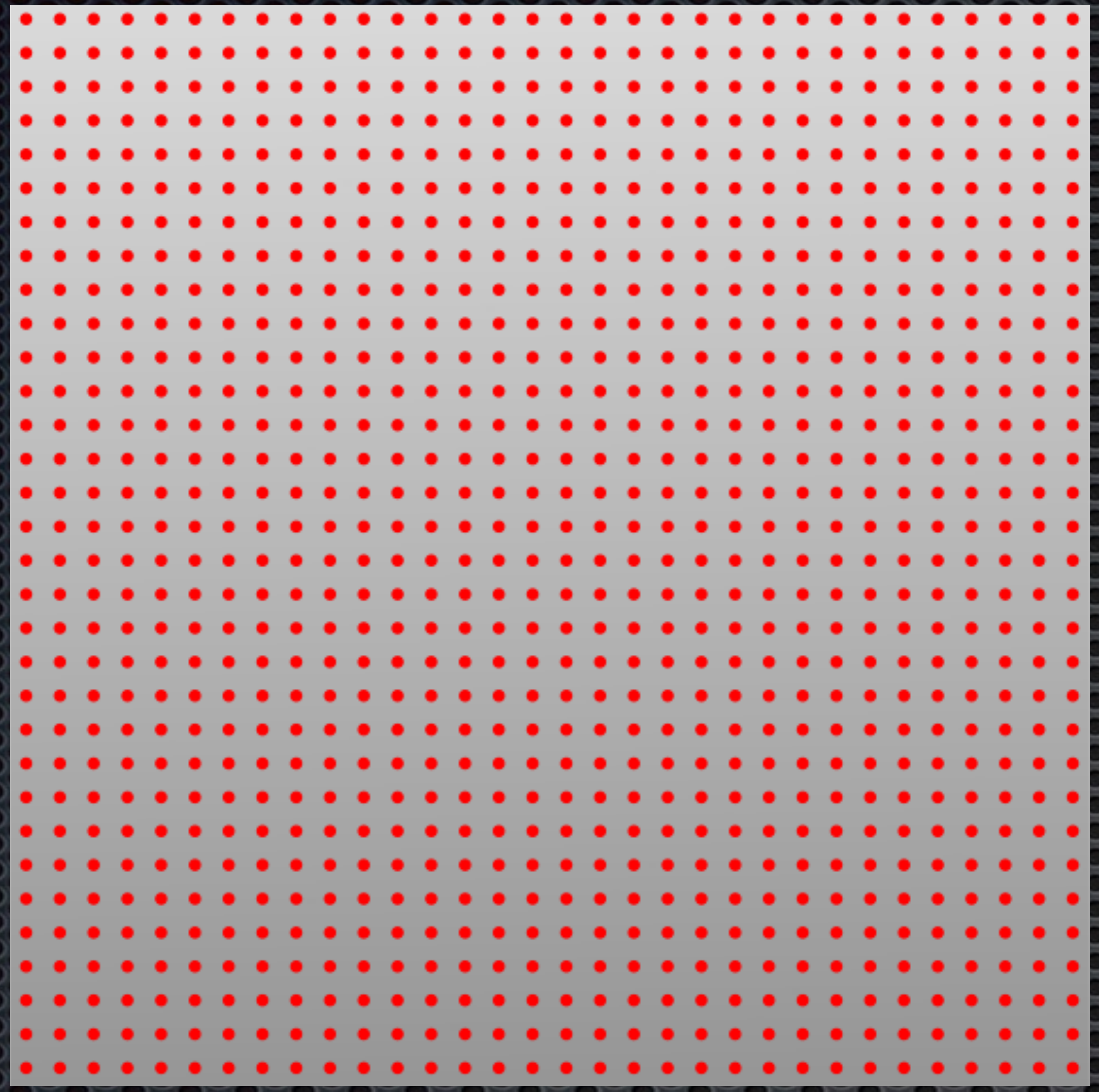


Light Simulation: Participating Media

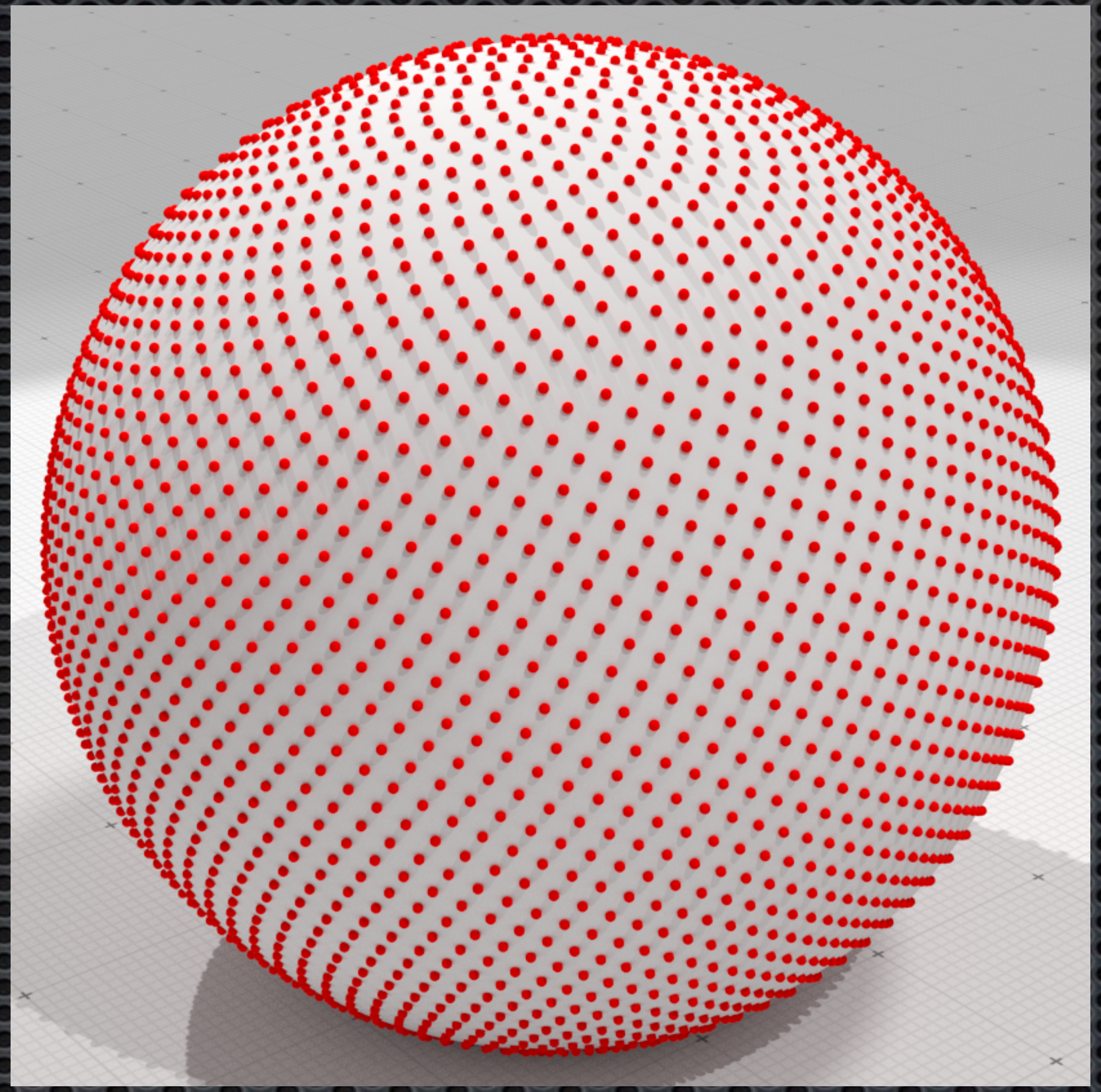


Regular Sampling Pattern

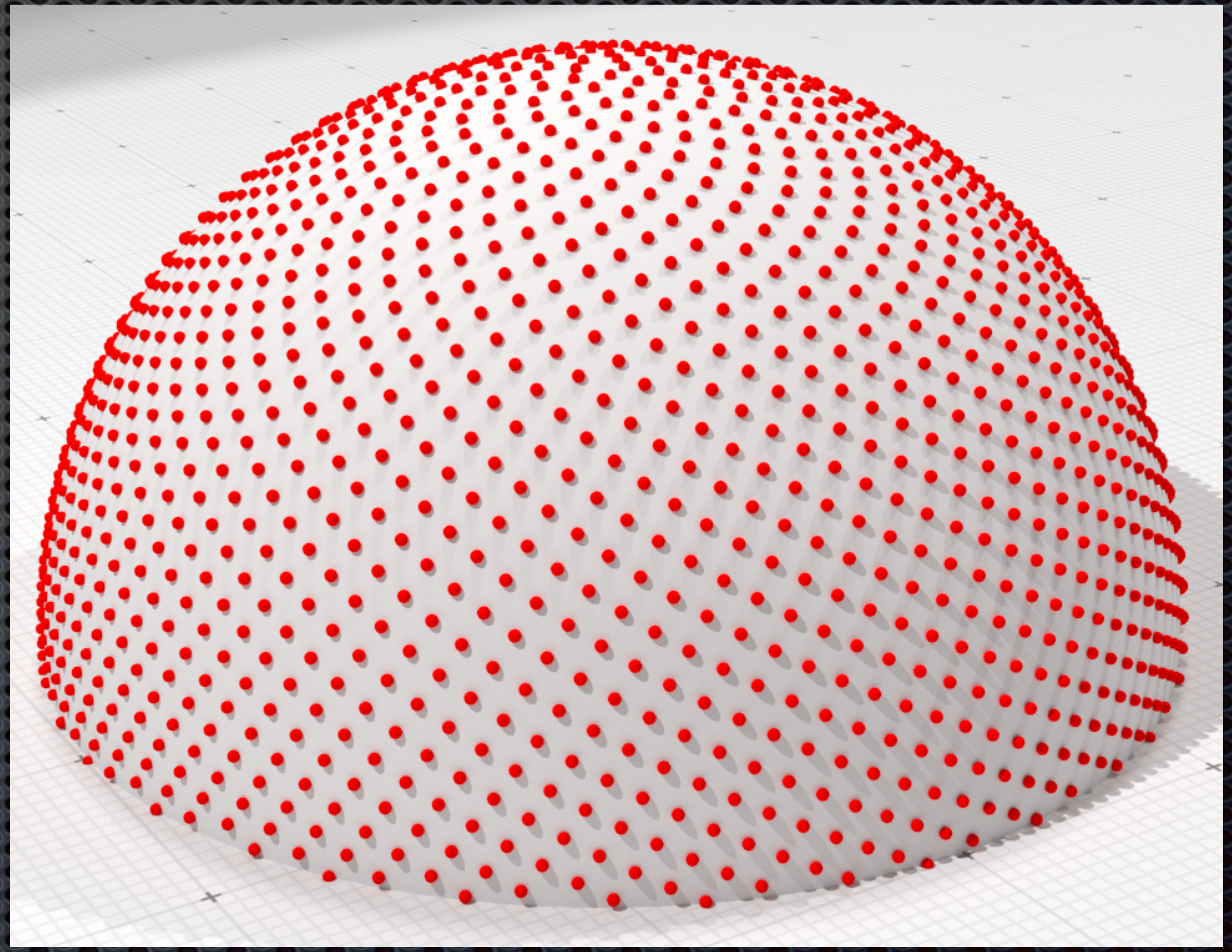
Euclidean



Spherical



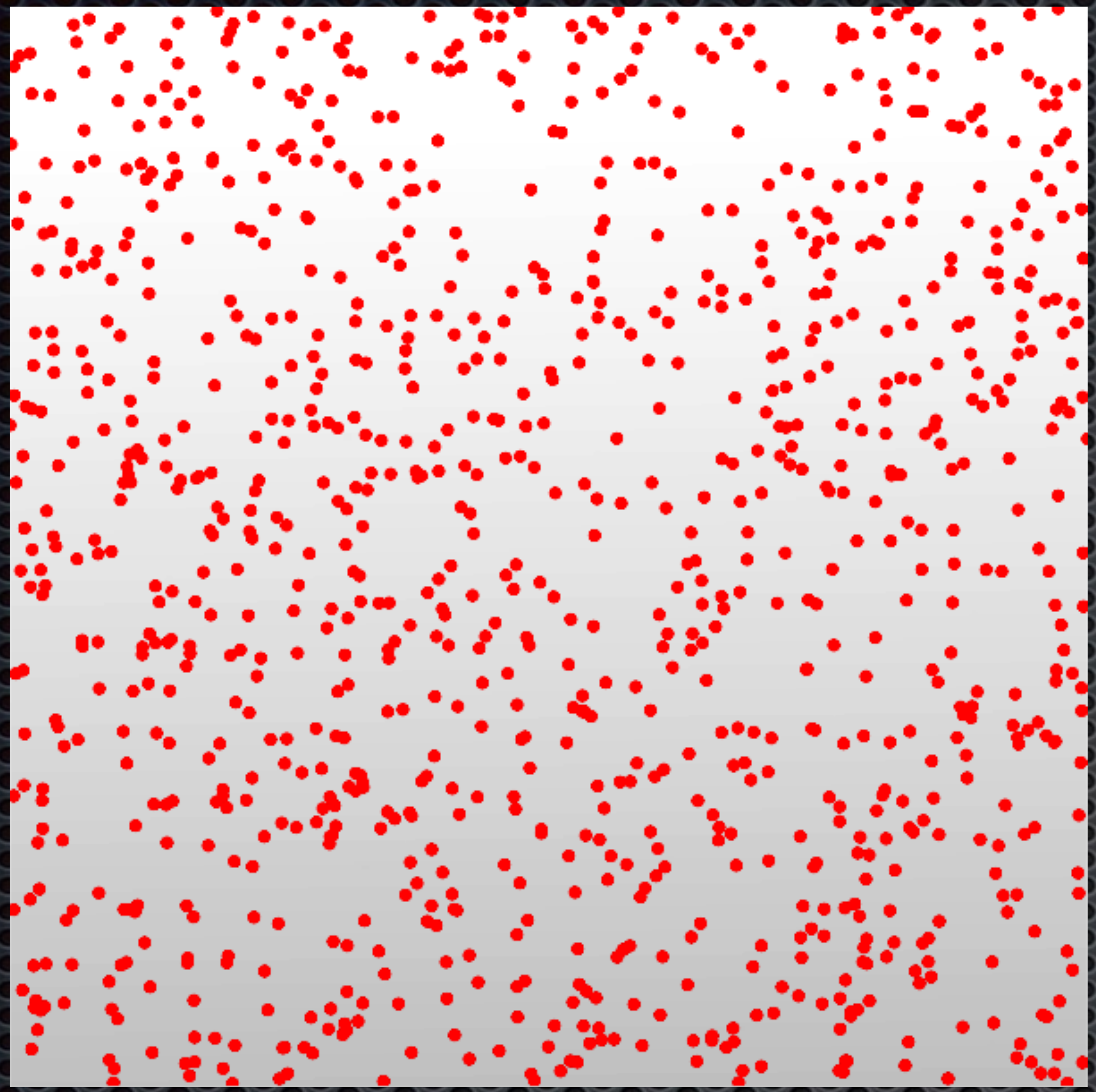
Hemispherical



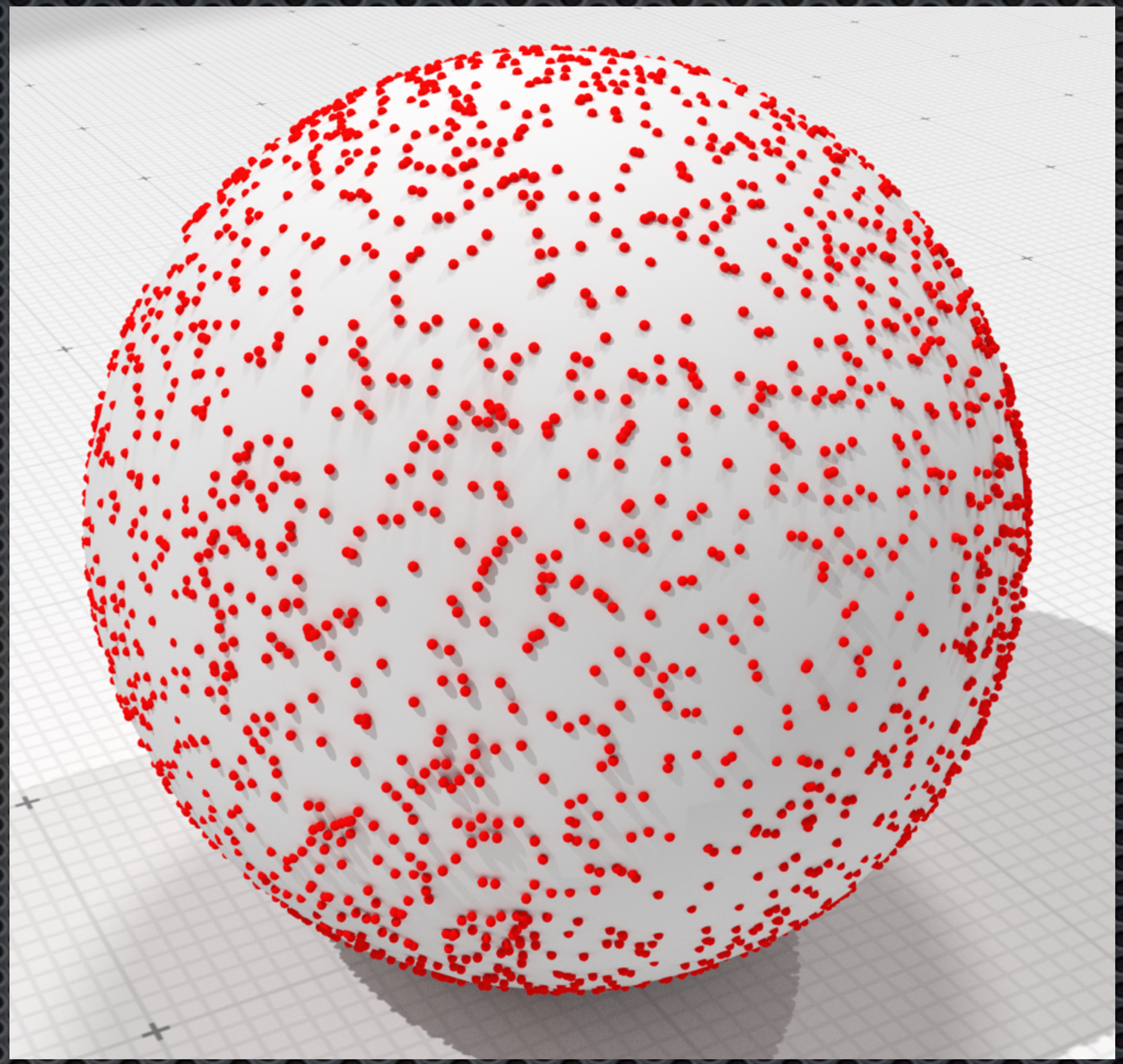
[Marques et al. 2013]

Purely Random Sampling Pattern

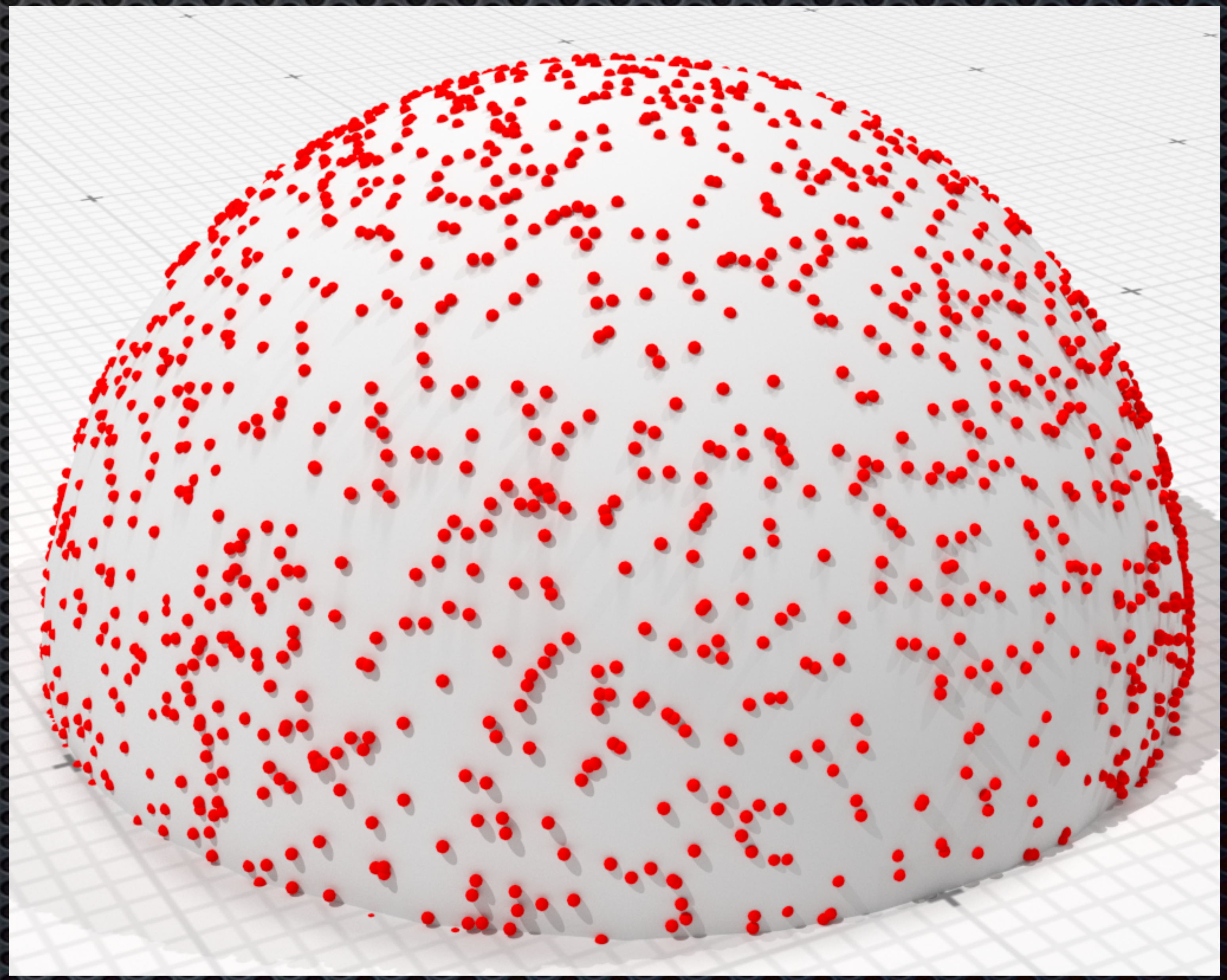
Euclidean



Spherical

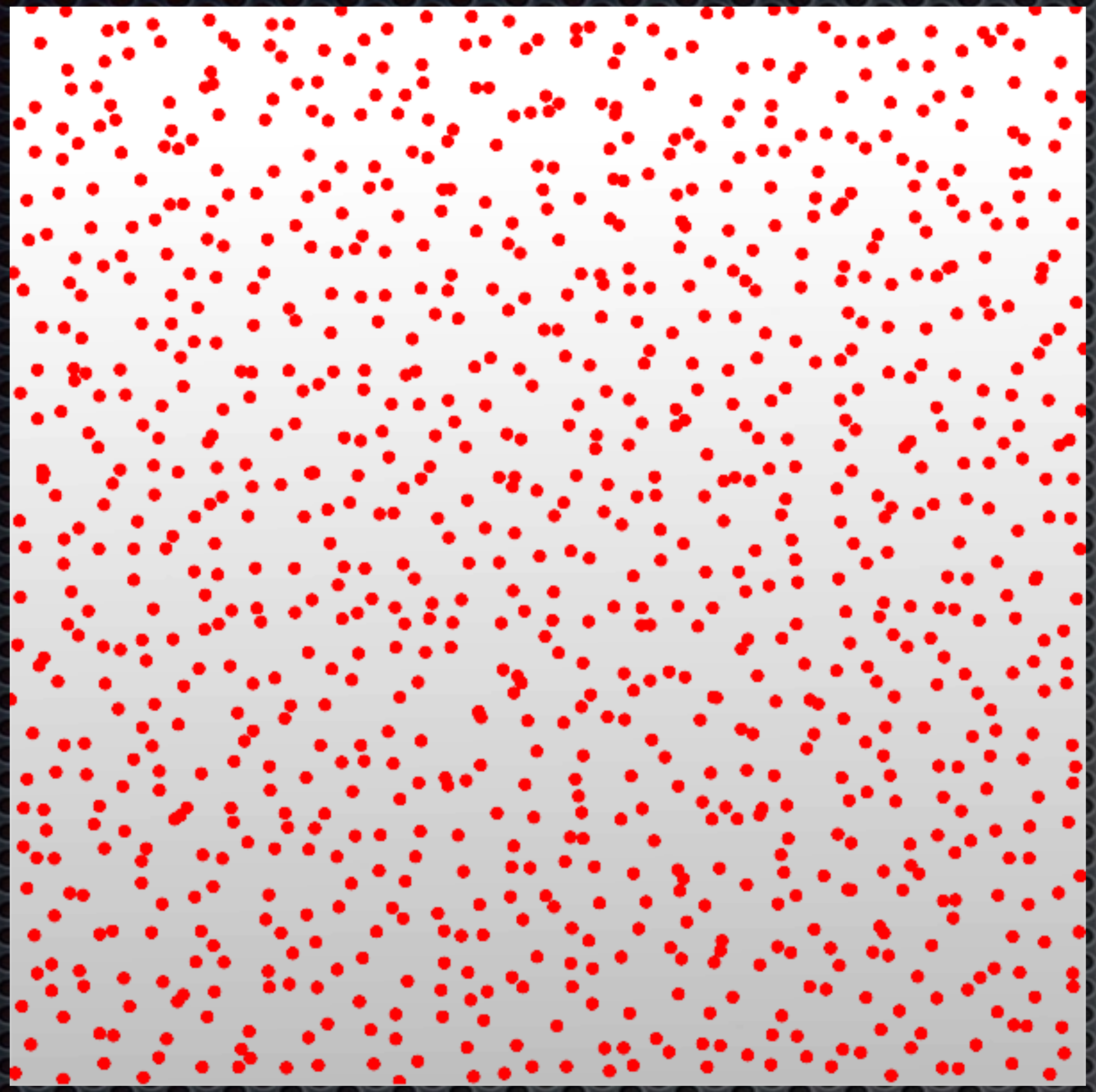


Hemispherical

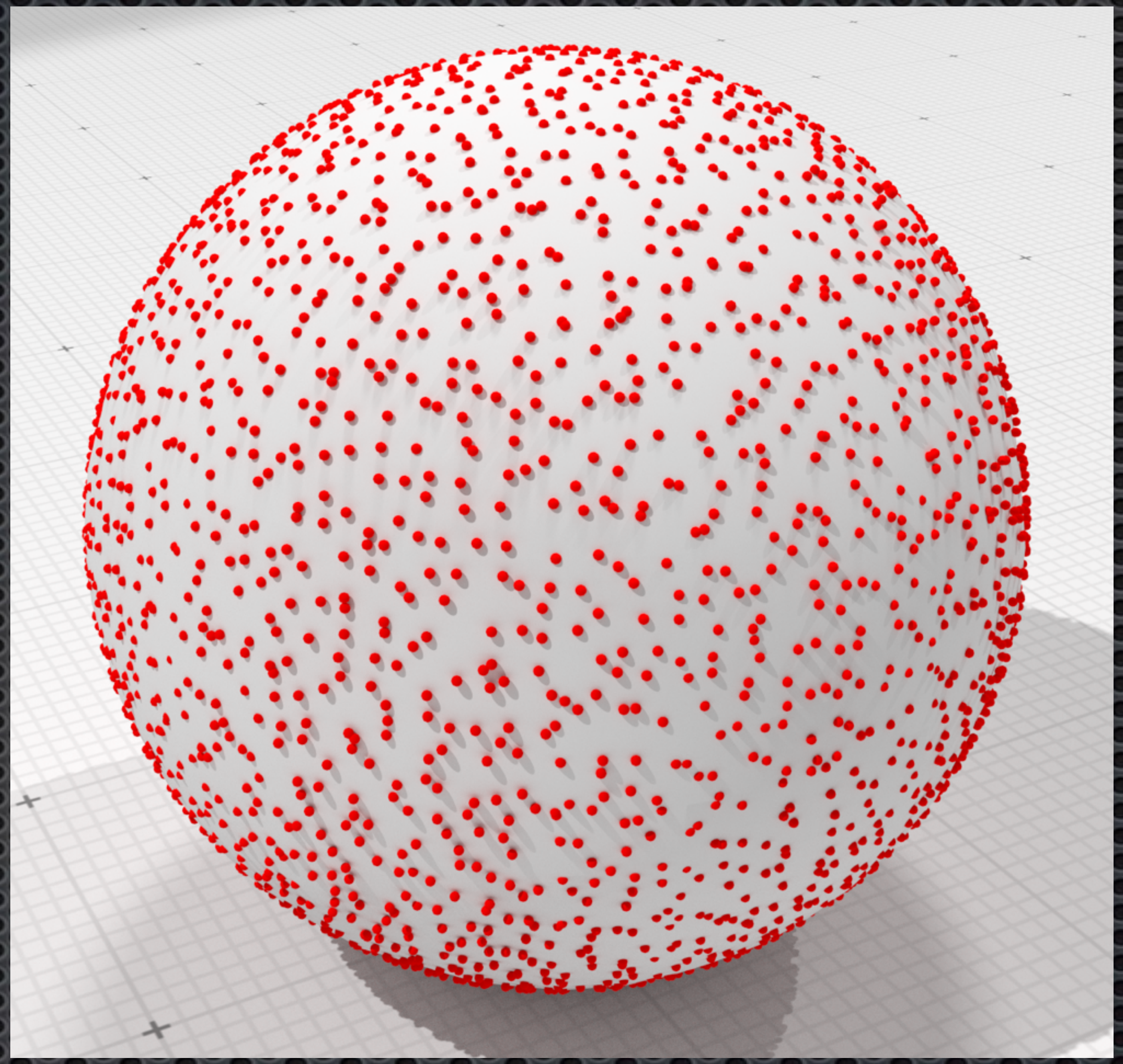


Jittered Sampling Pattern

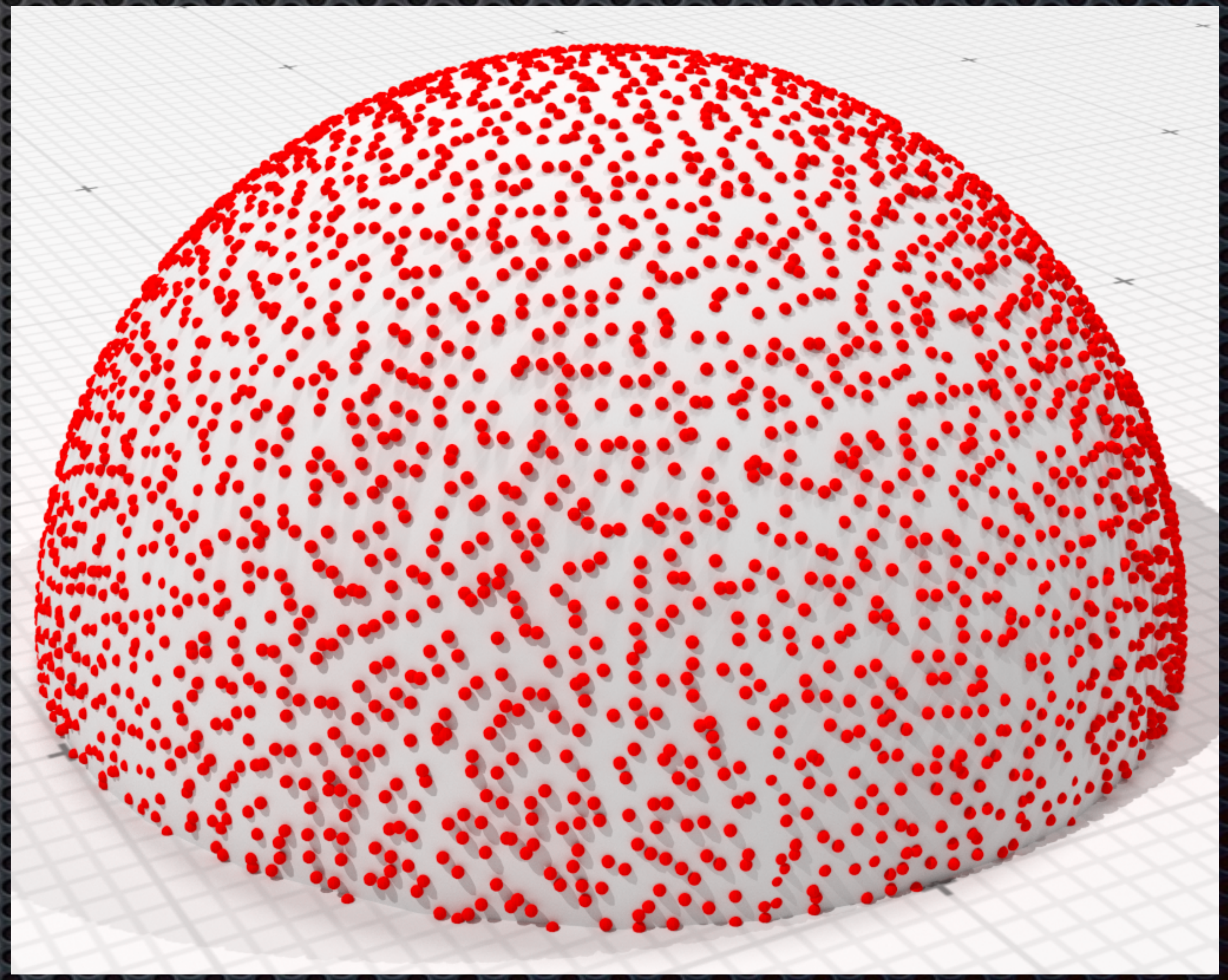
Euclidean



Spherical

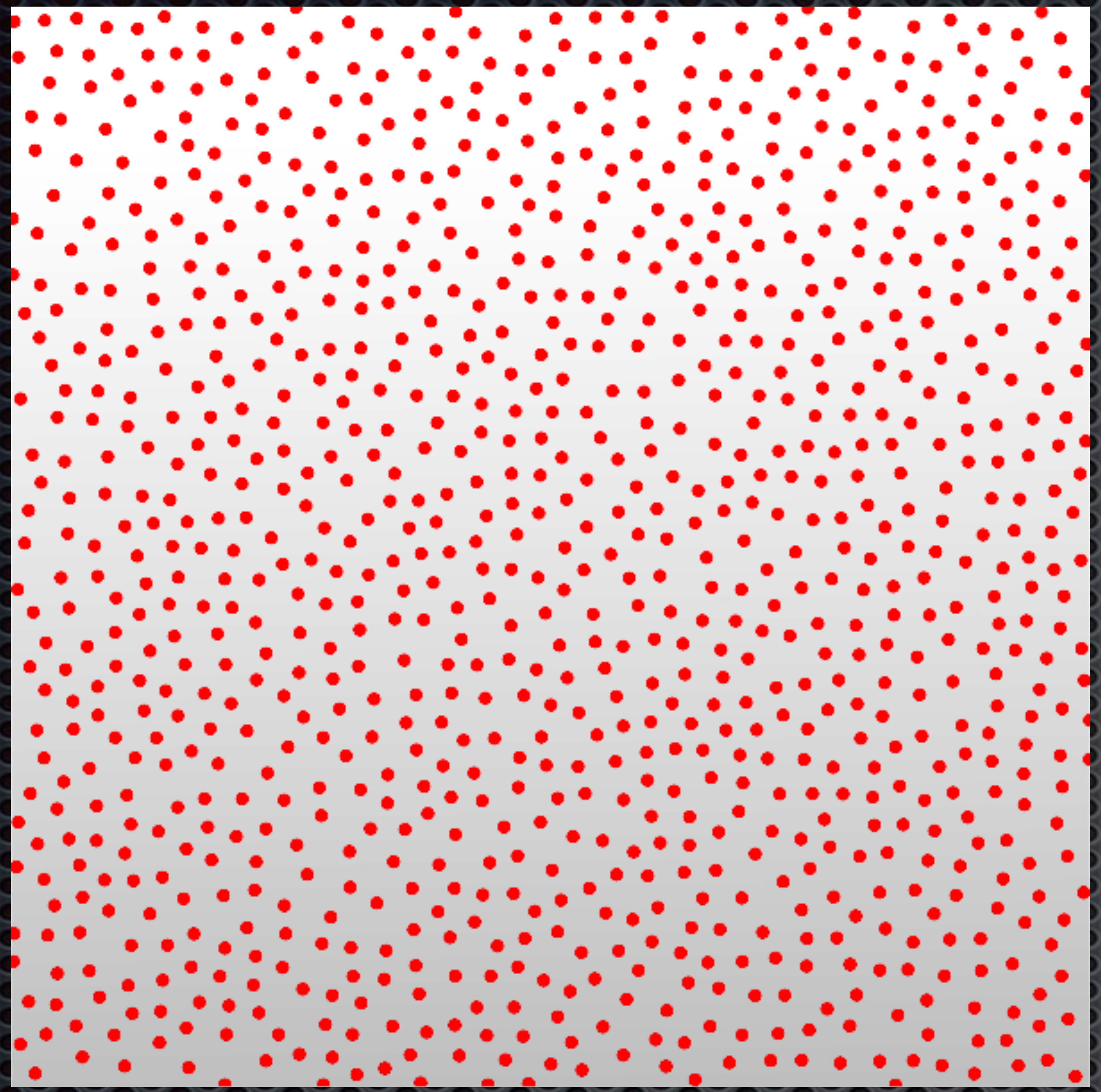


Hemispherical

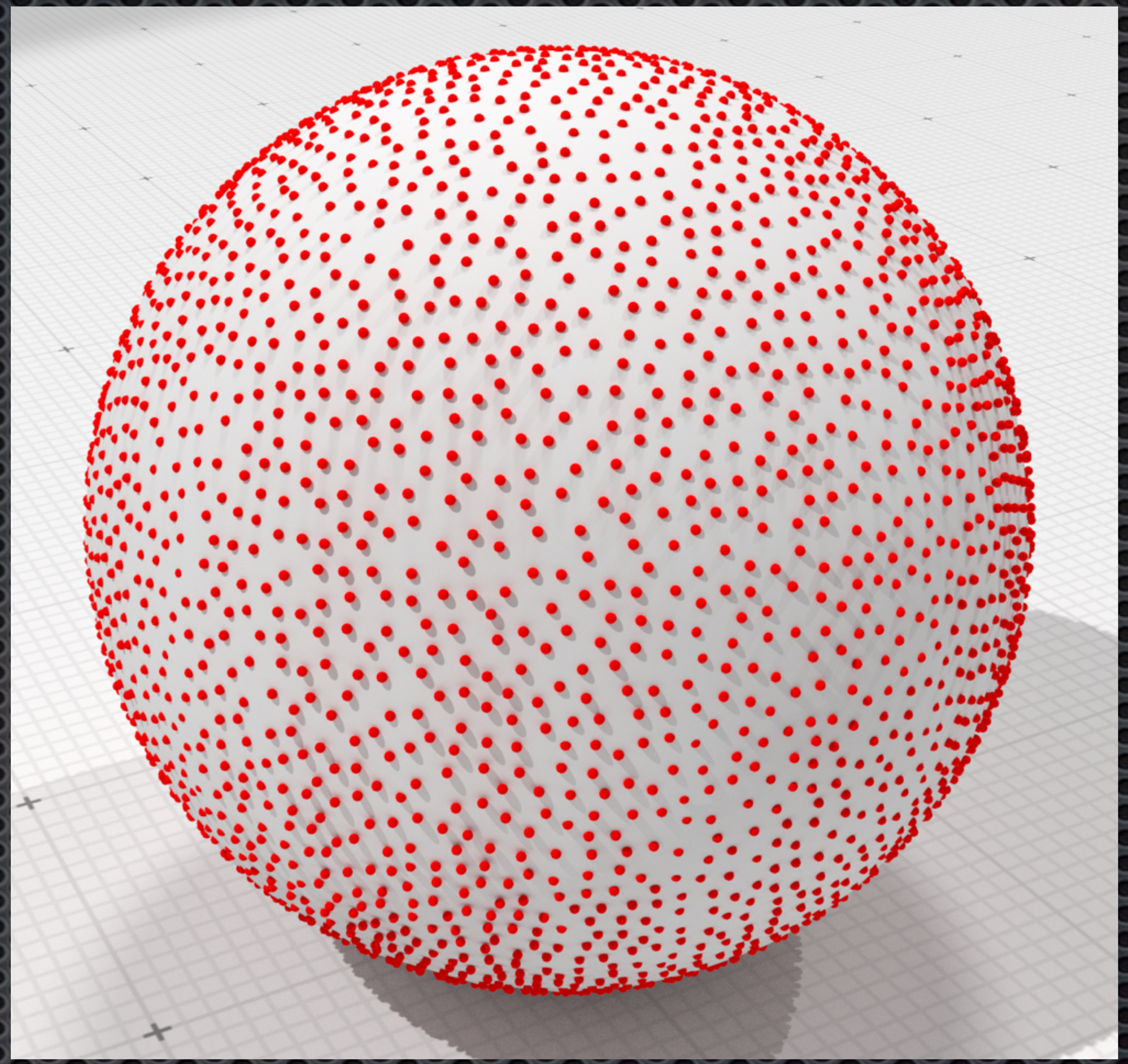


Poisson Disk Sampling Pattern

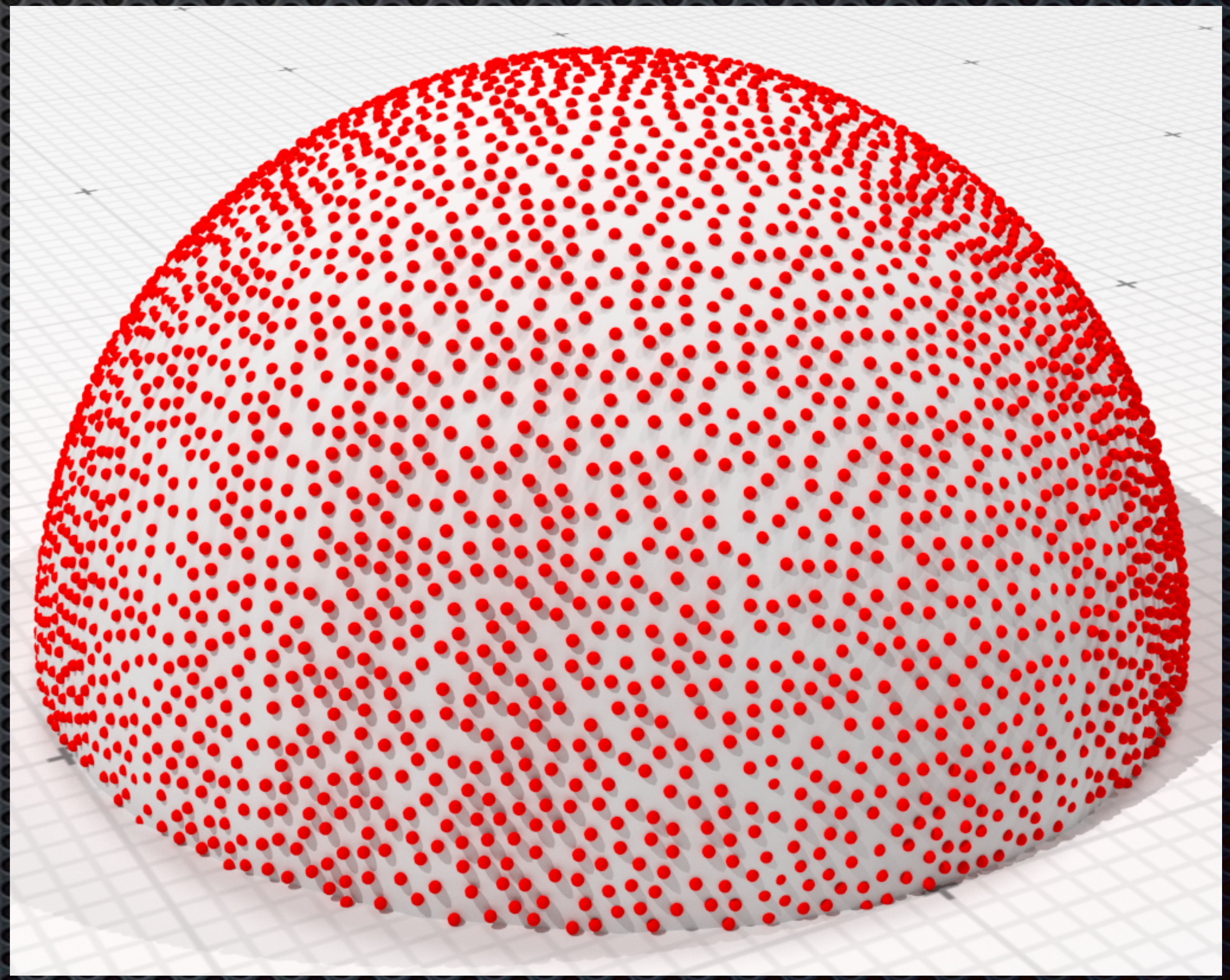
Euclidean



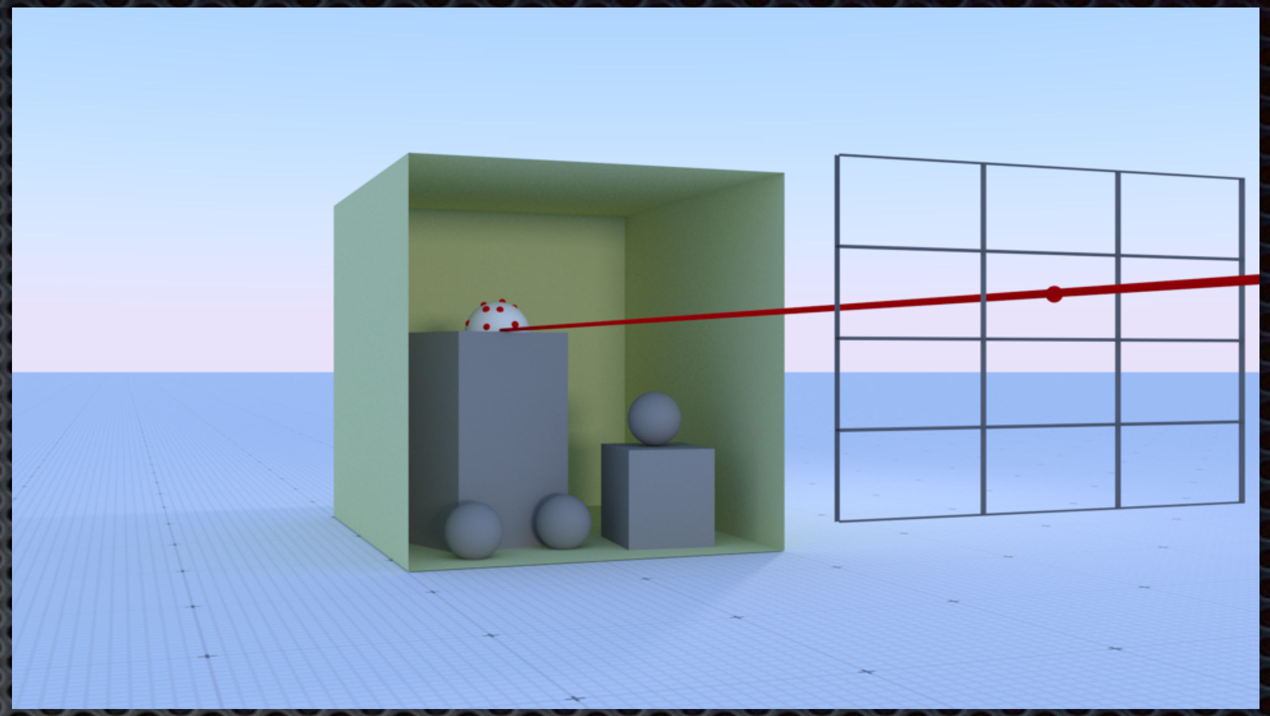
Spherical

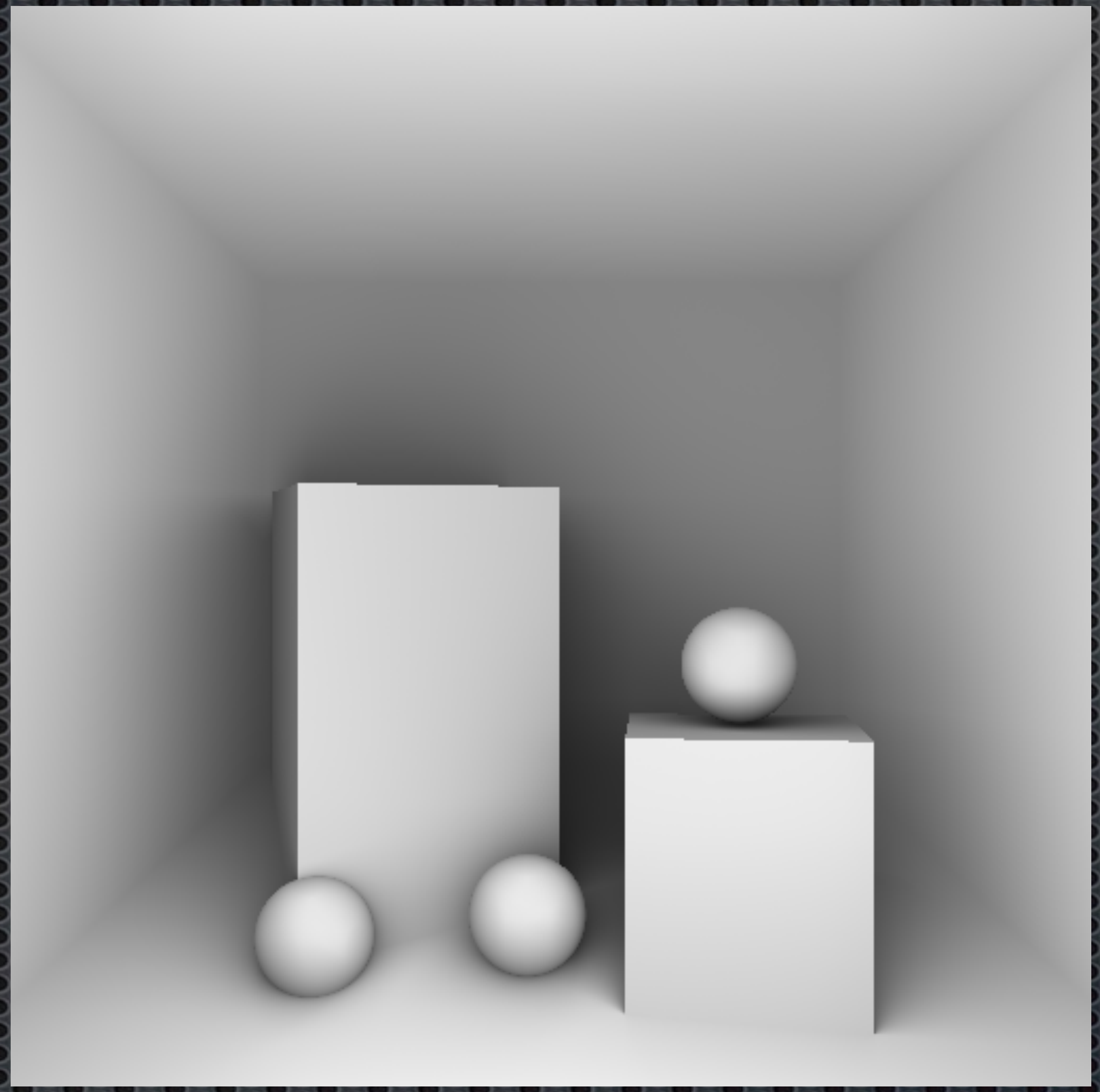


Hemispherical



Ambient Occlusion





Geometric Aliasing

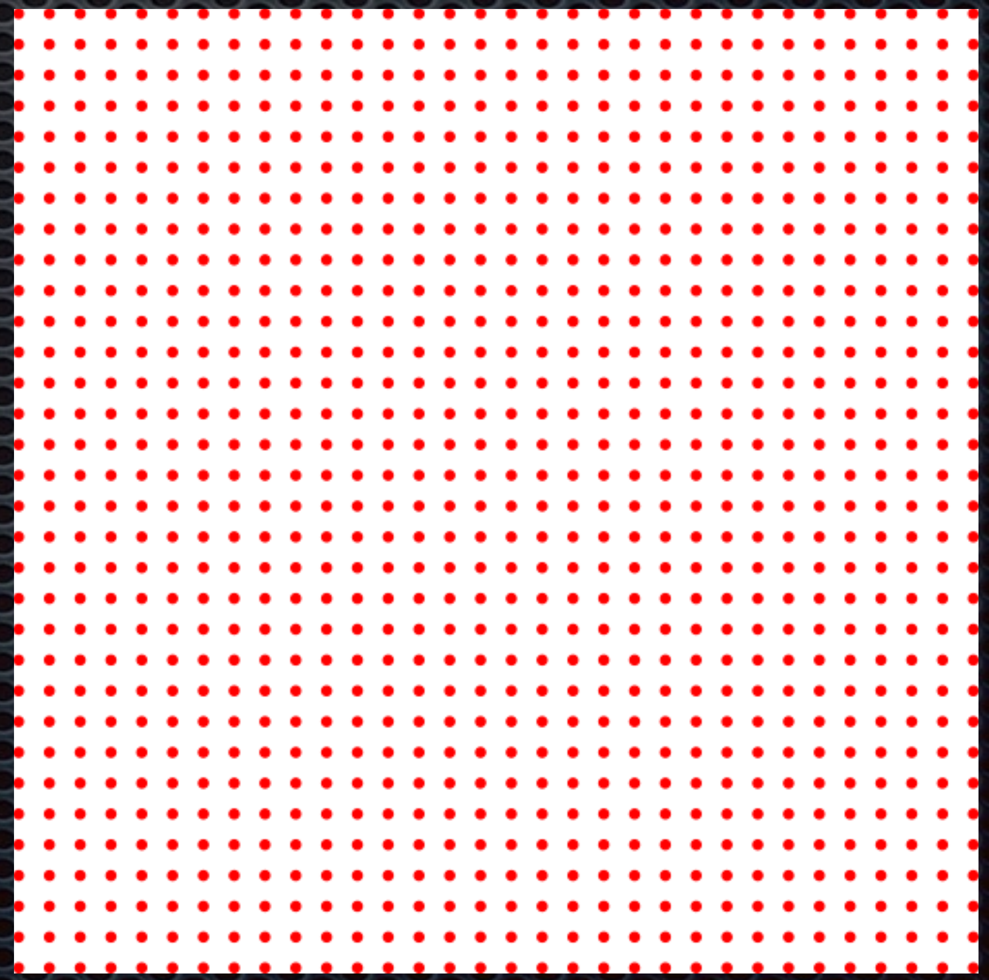
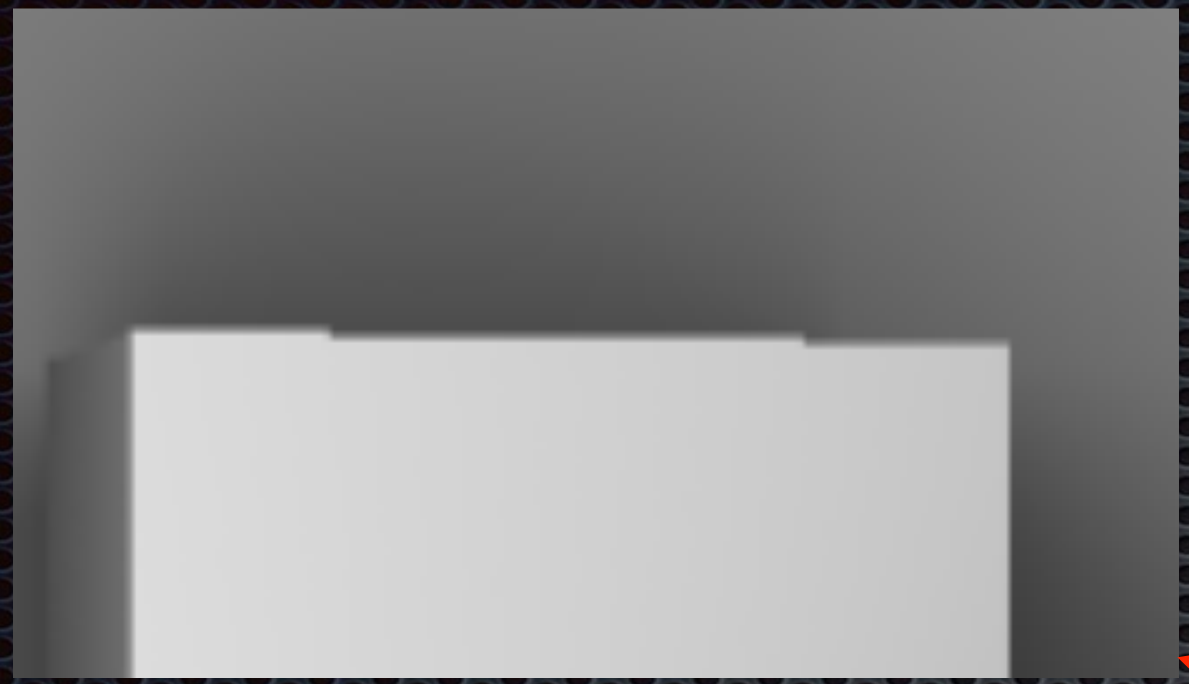
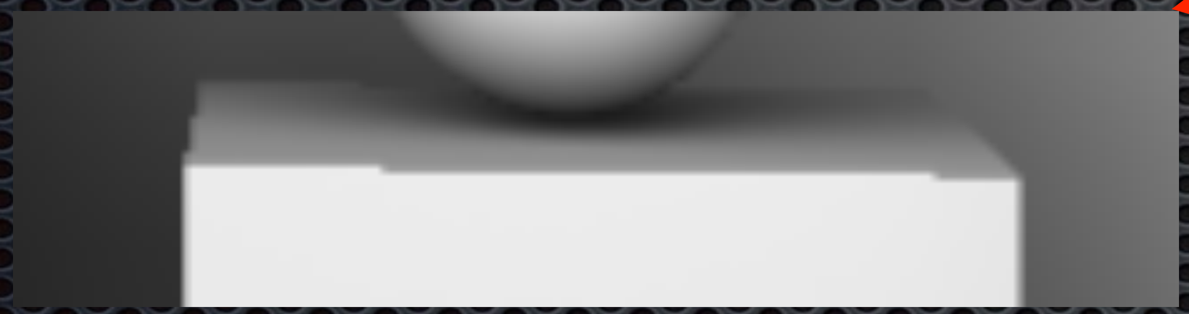
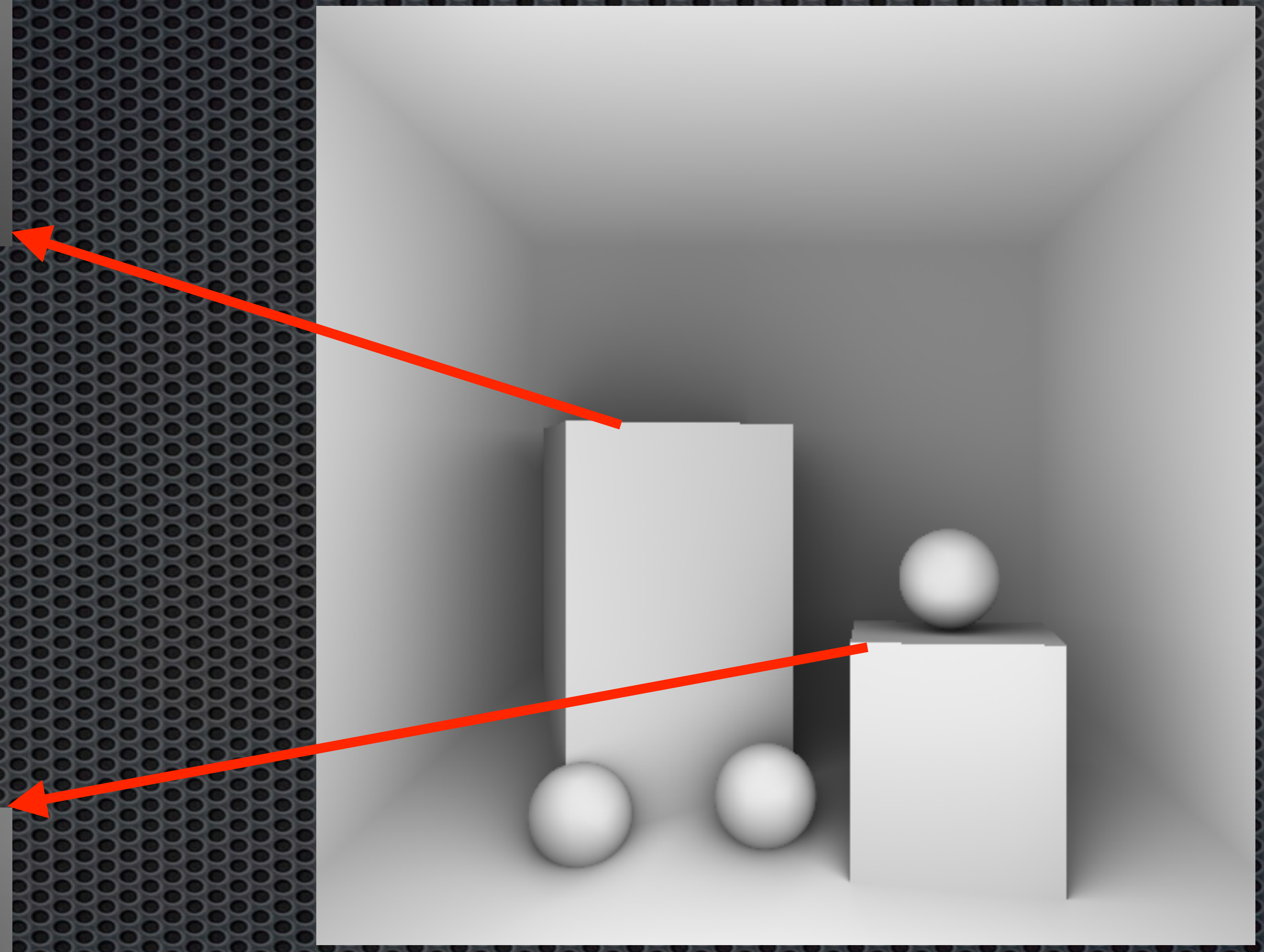


Image Plane

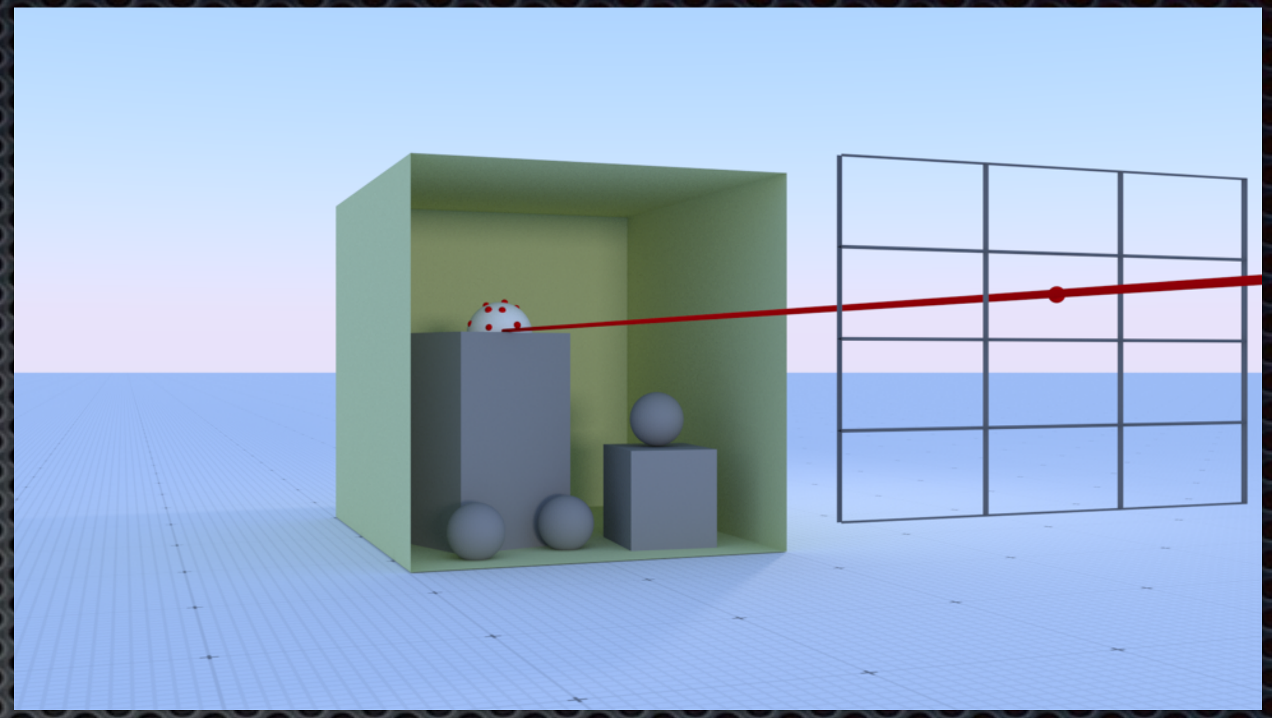
Ambient Occlusion







Geometric Aliasing



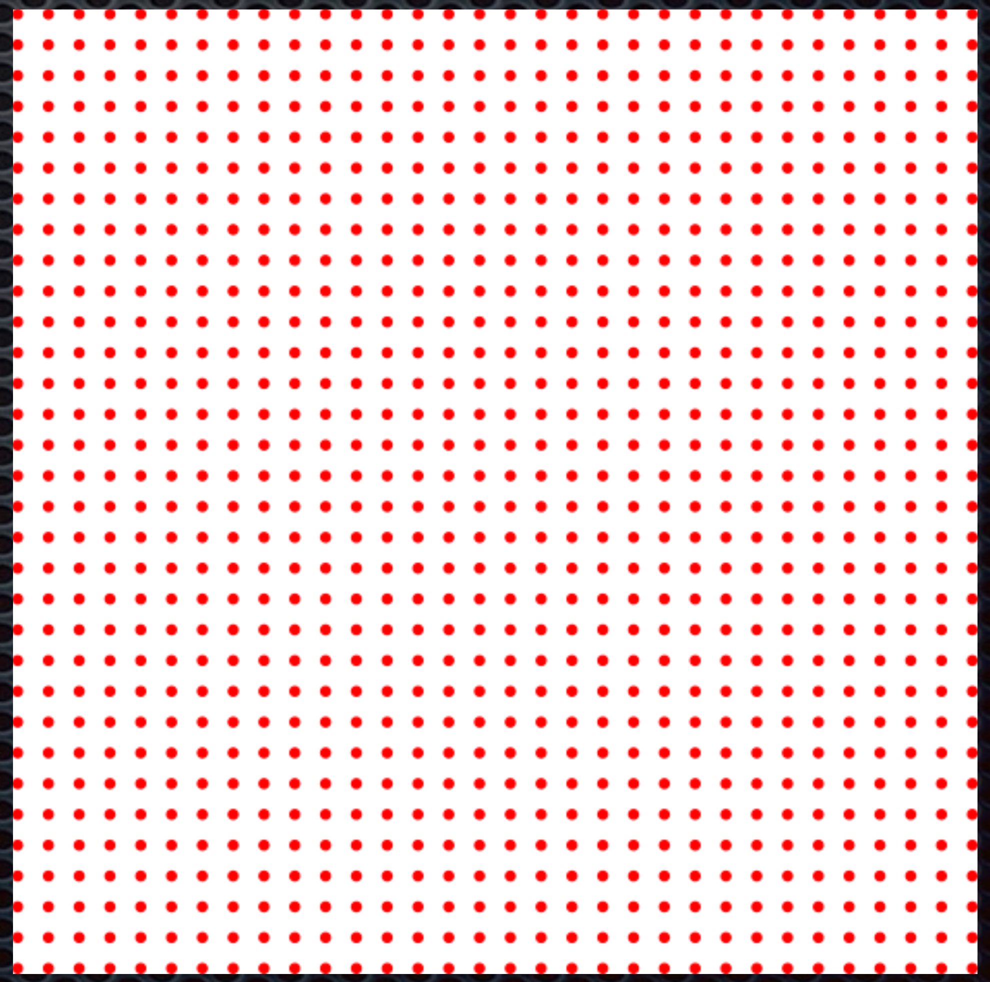
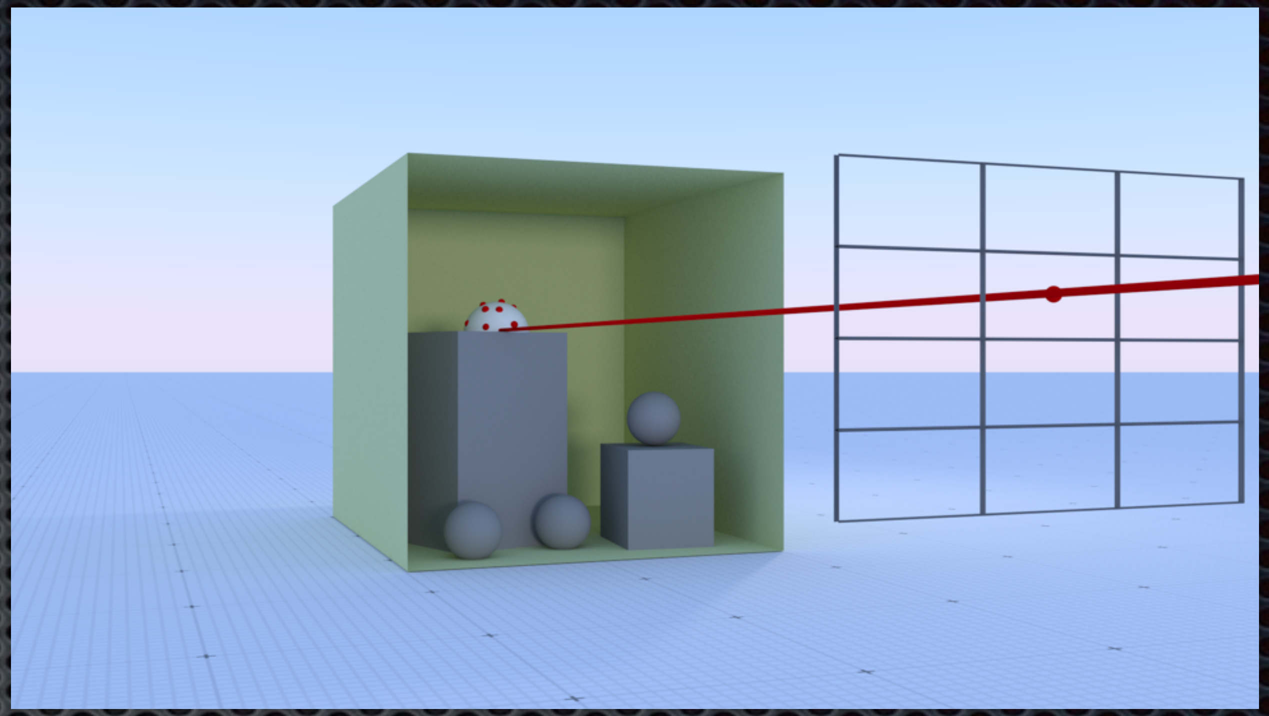
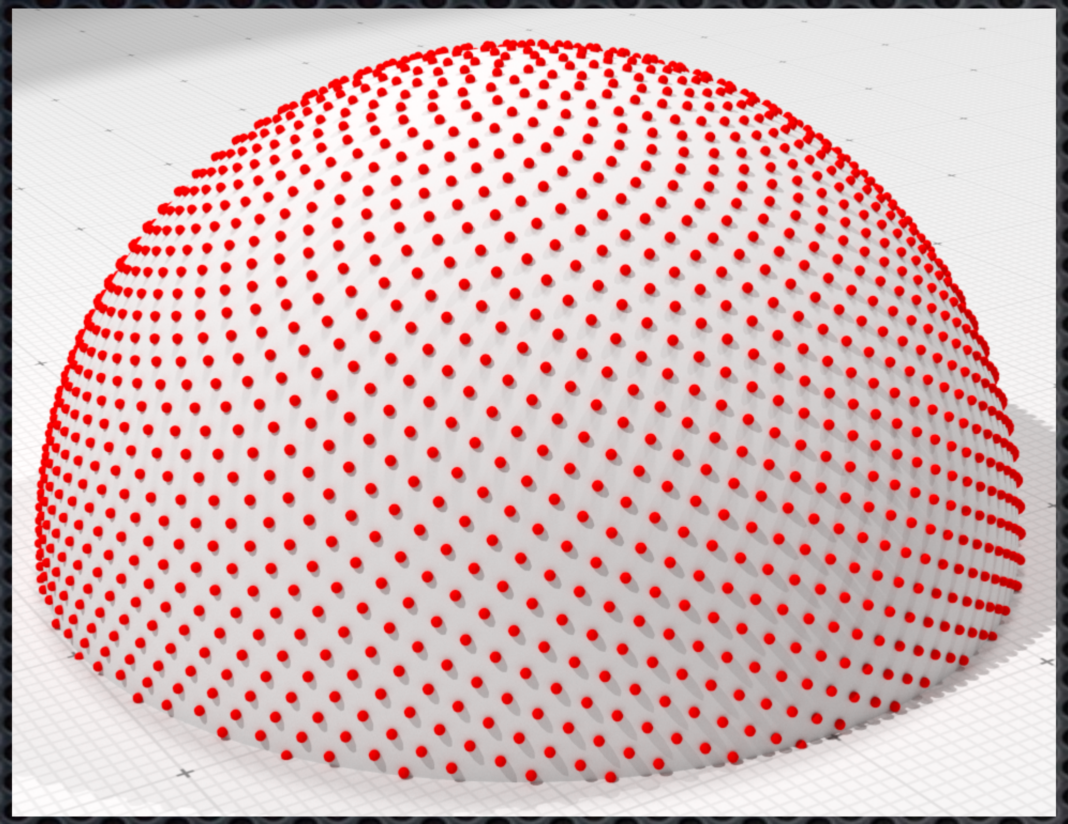


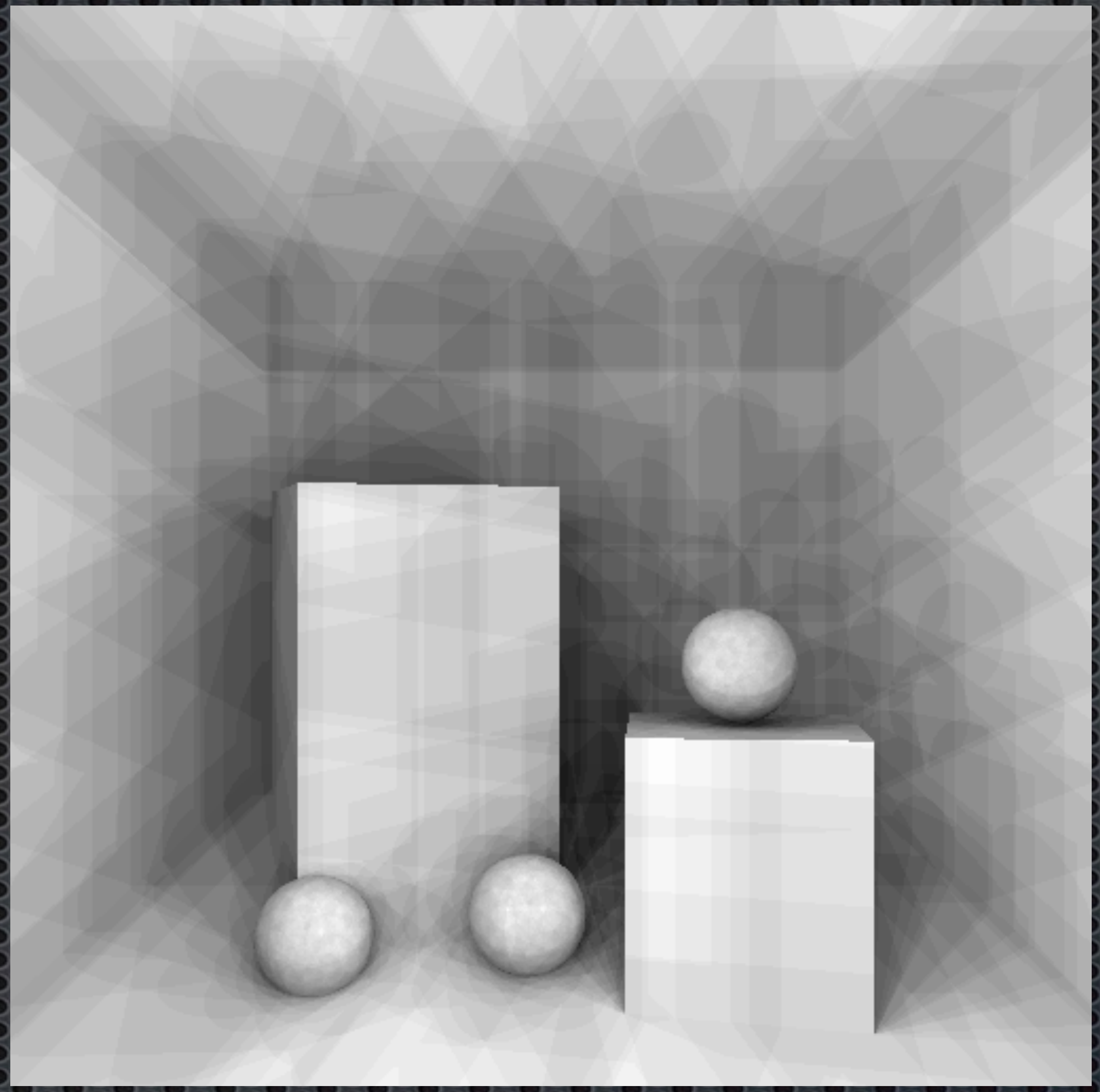
Image Plane

Error: Structure Artifacts





Hemisphere



Same Hemispherical pointset at all hitpoints

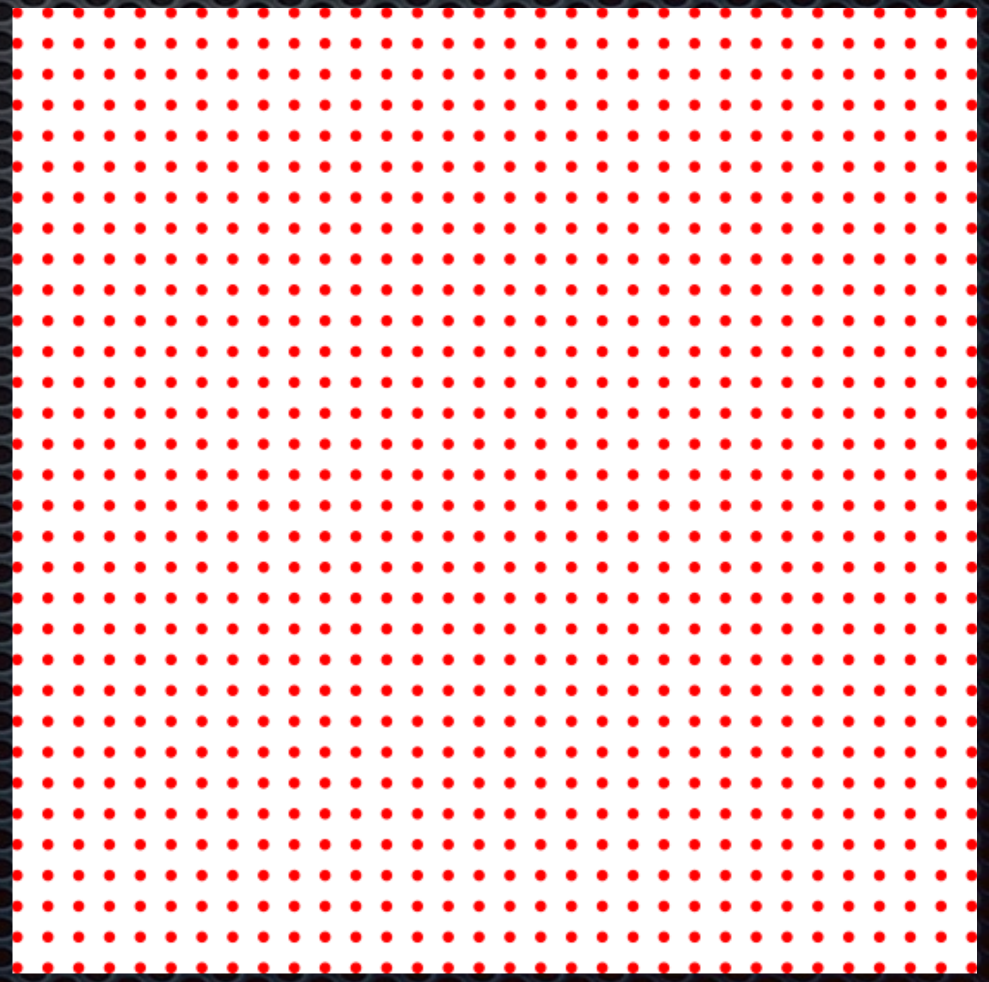
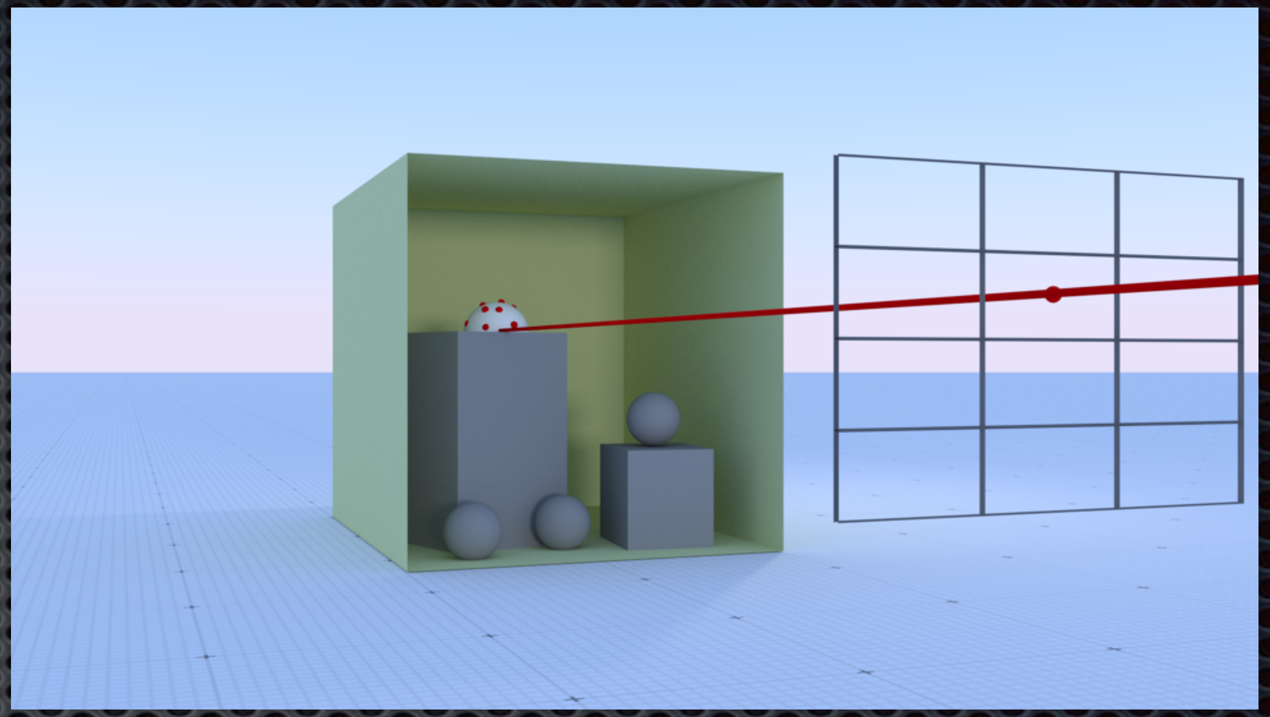
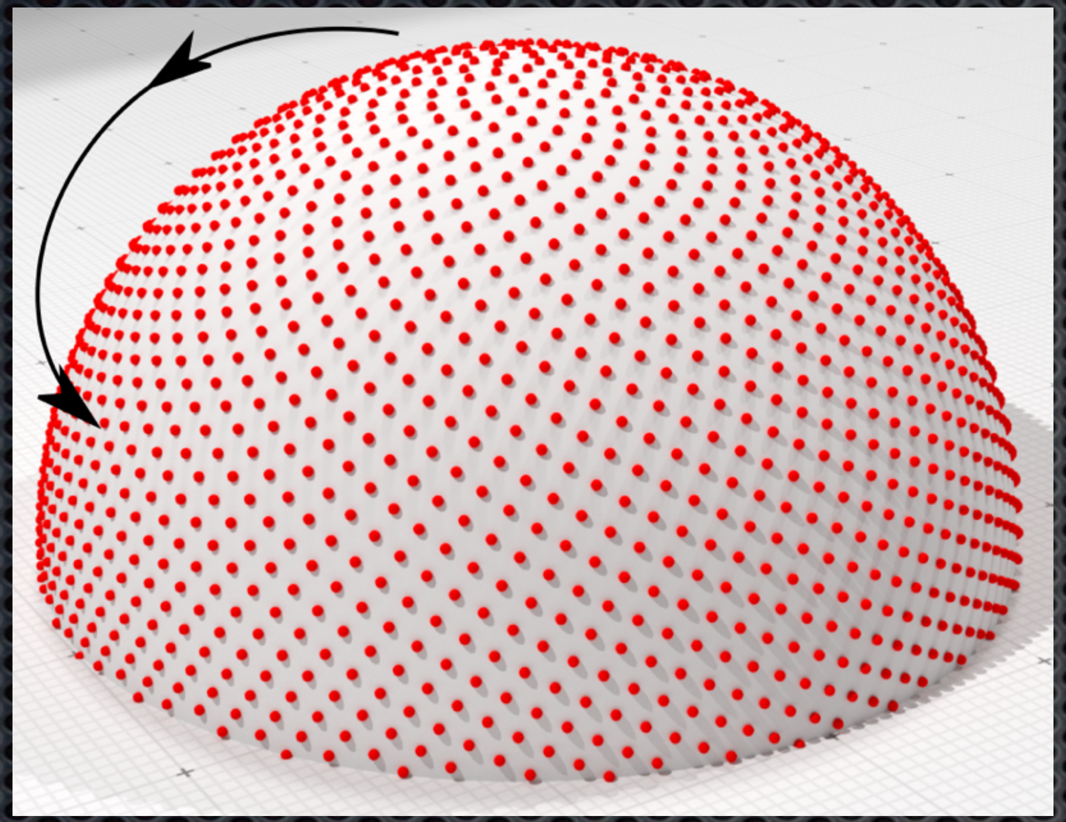
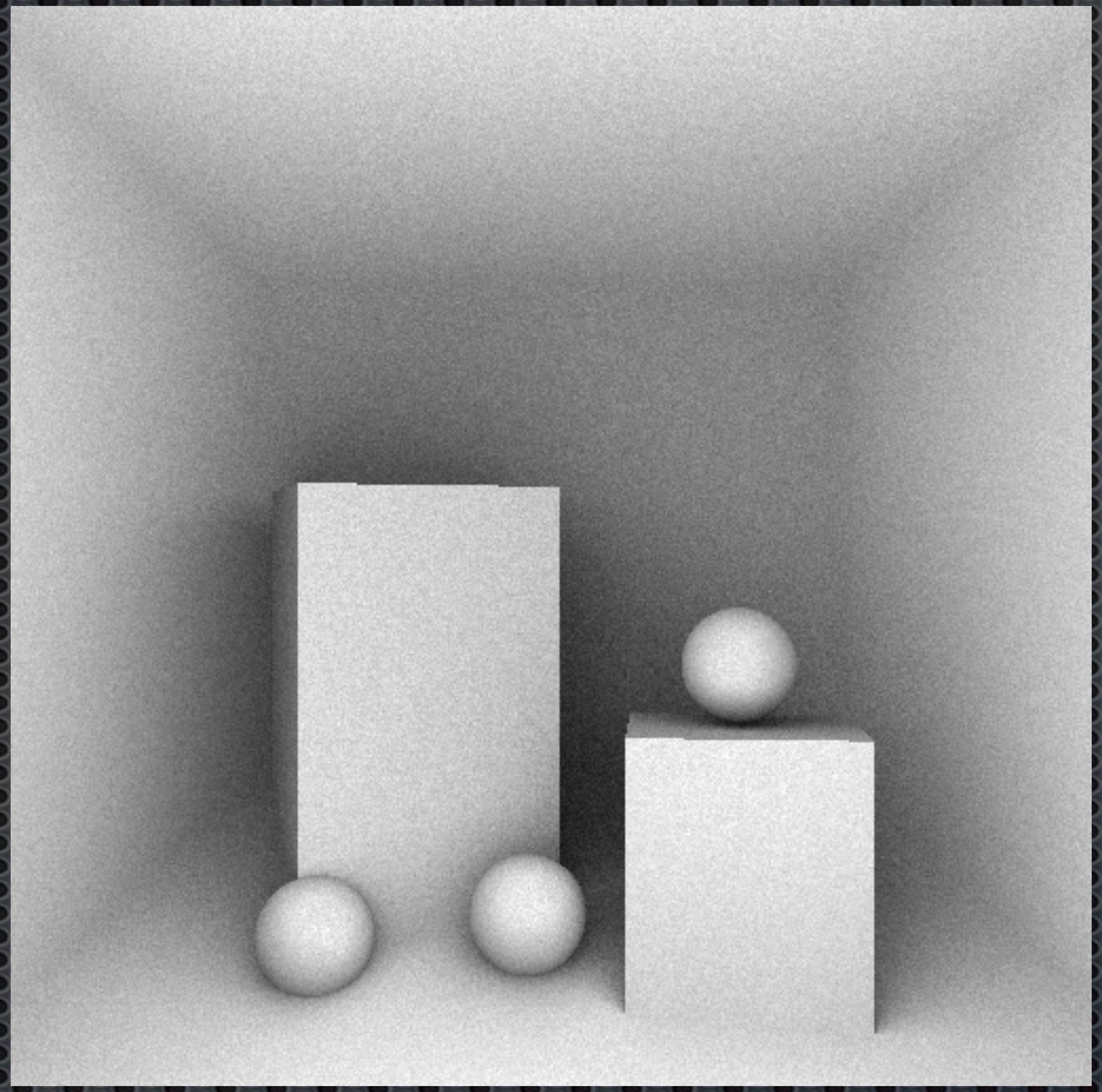


Image Plane

Error: Noise







With rotation

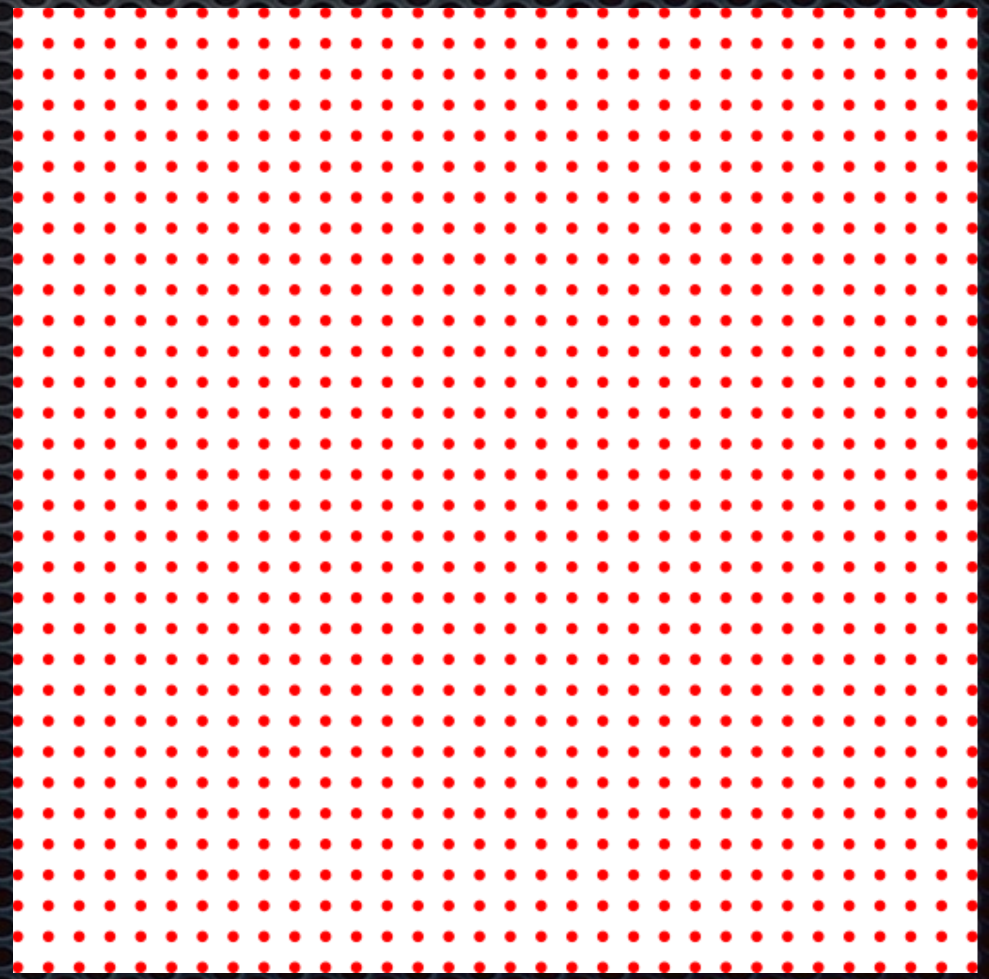
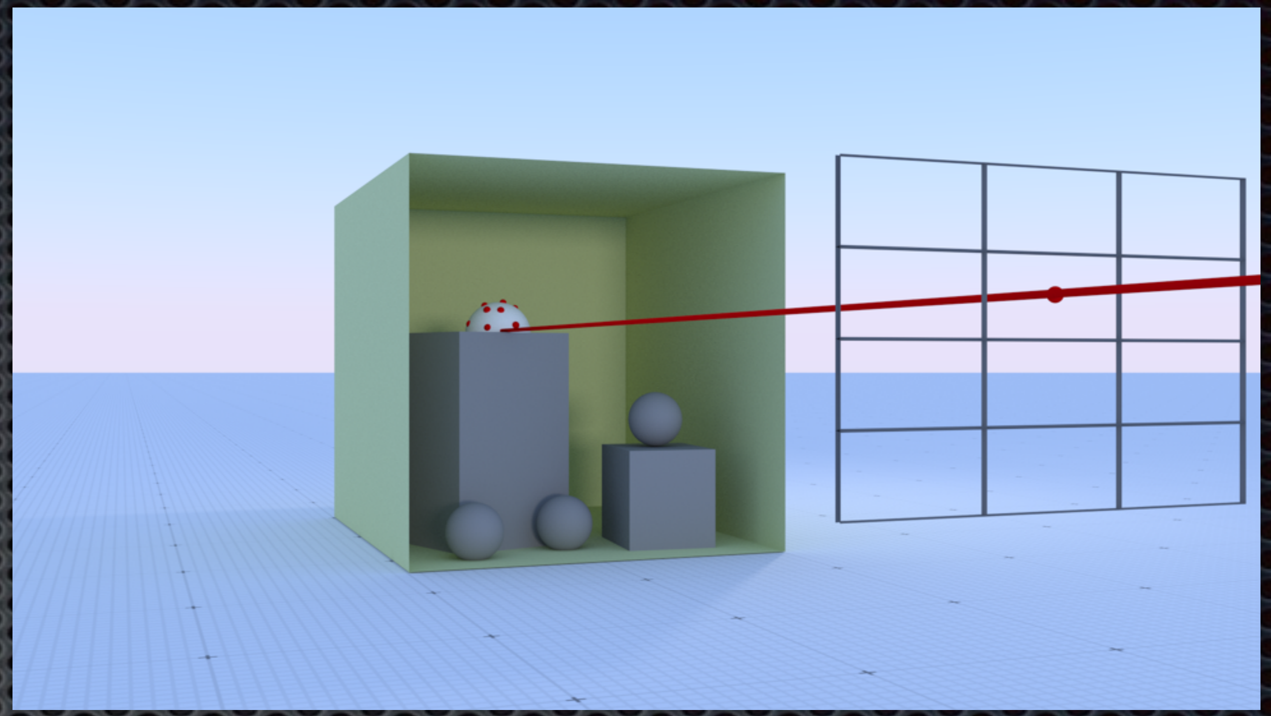
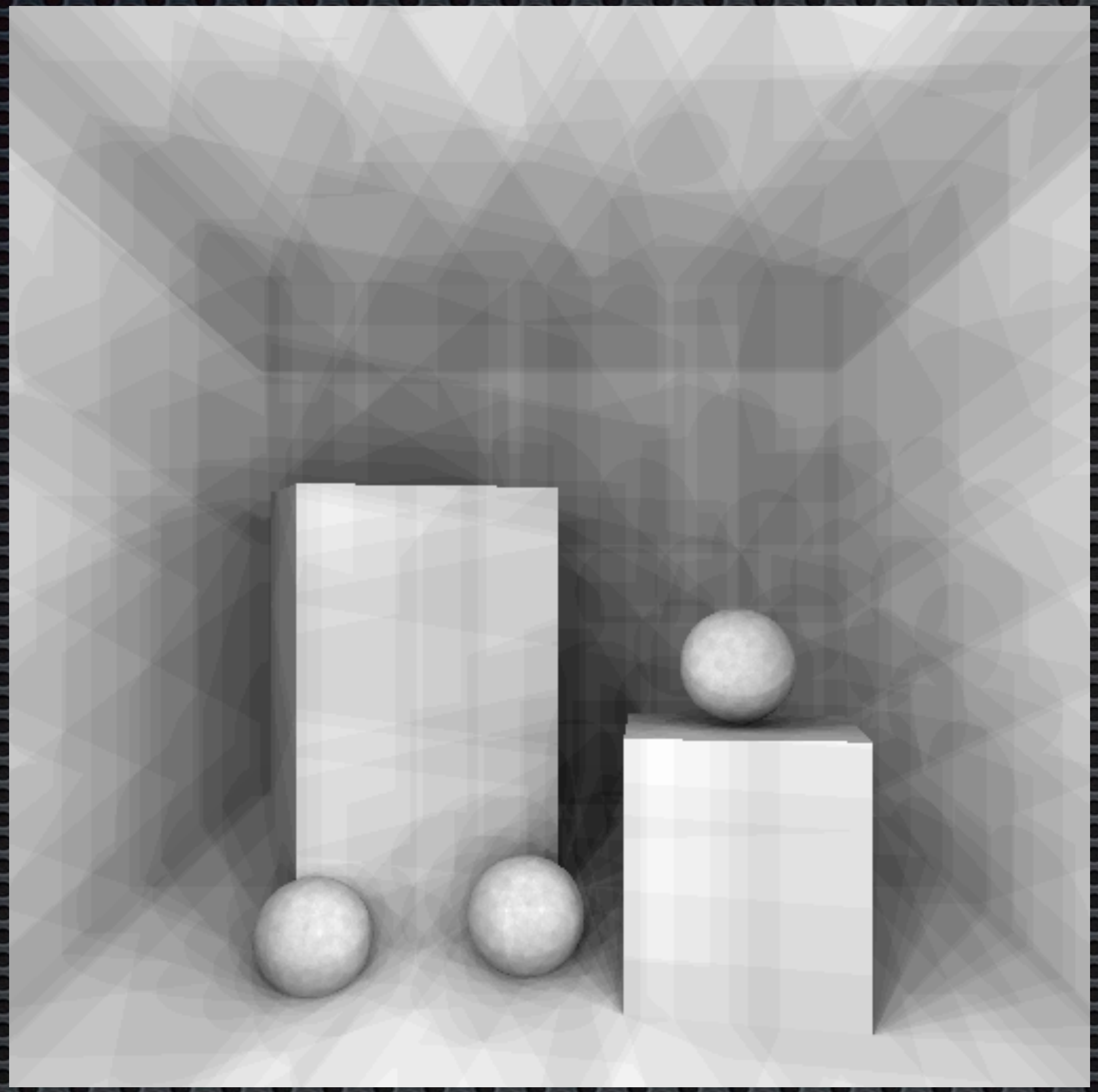


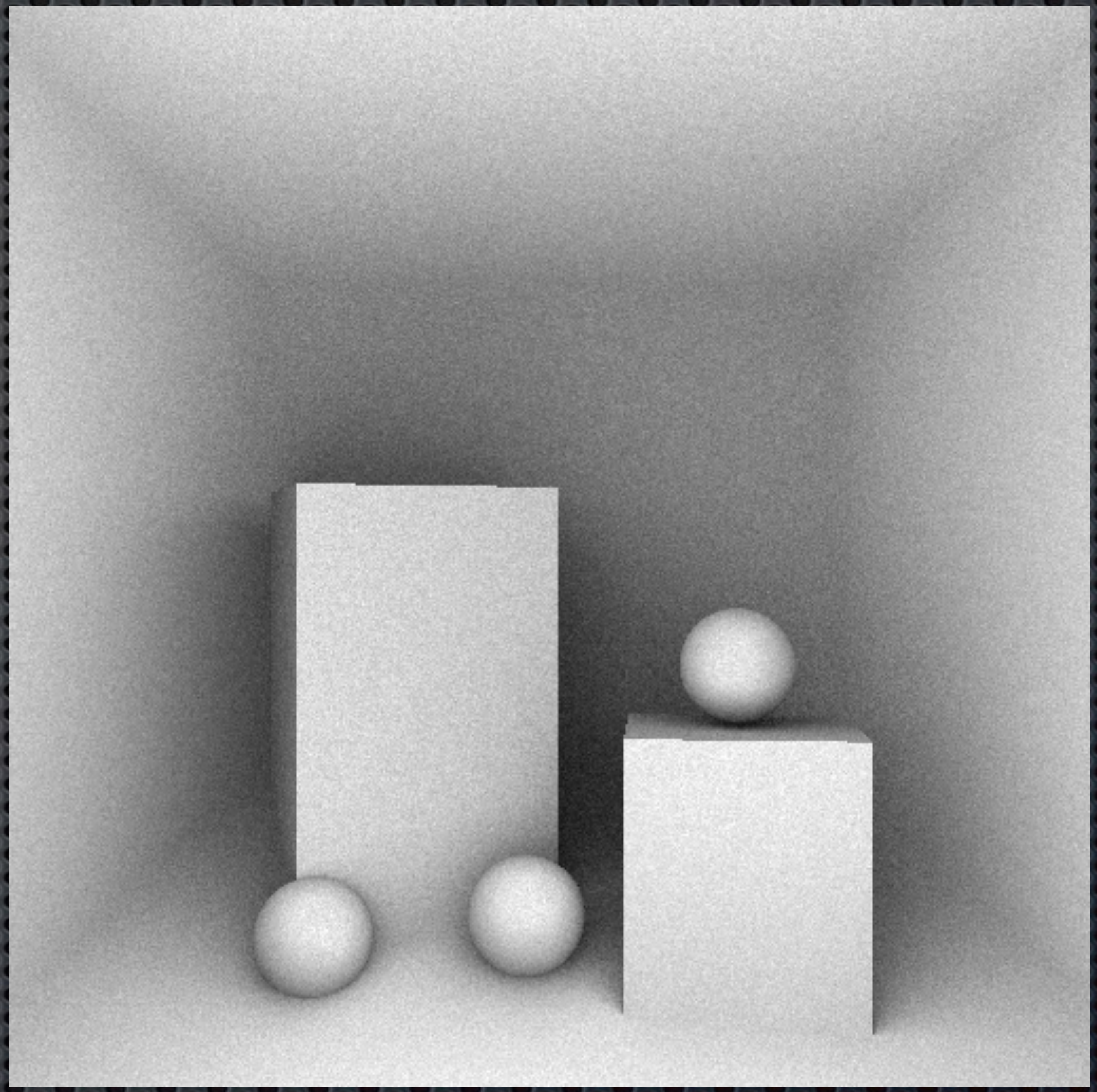
Image Plane

Error: Structures to Noise

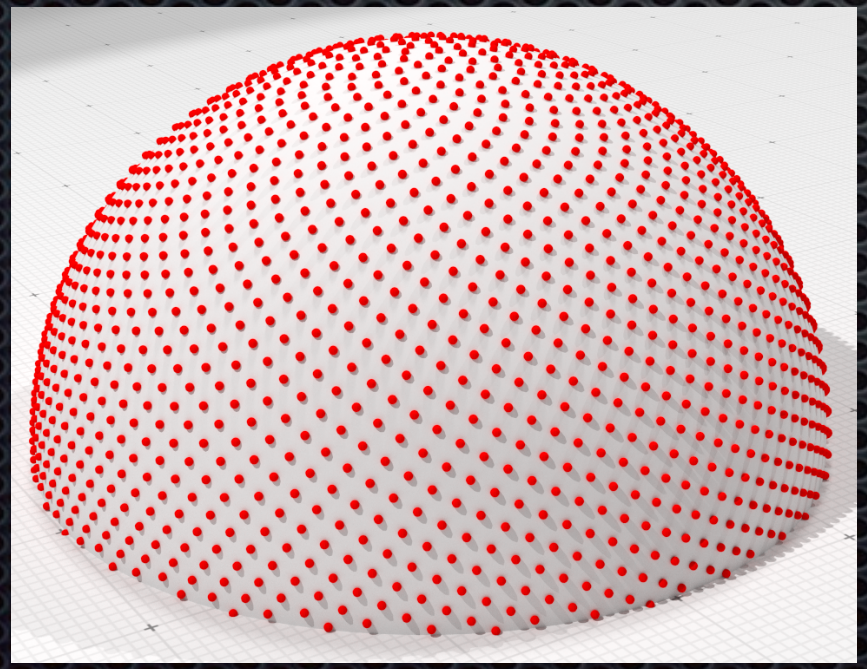


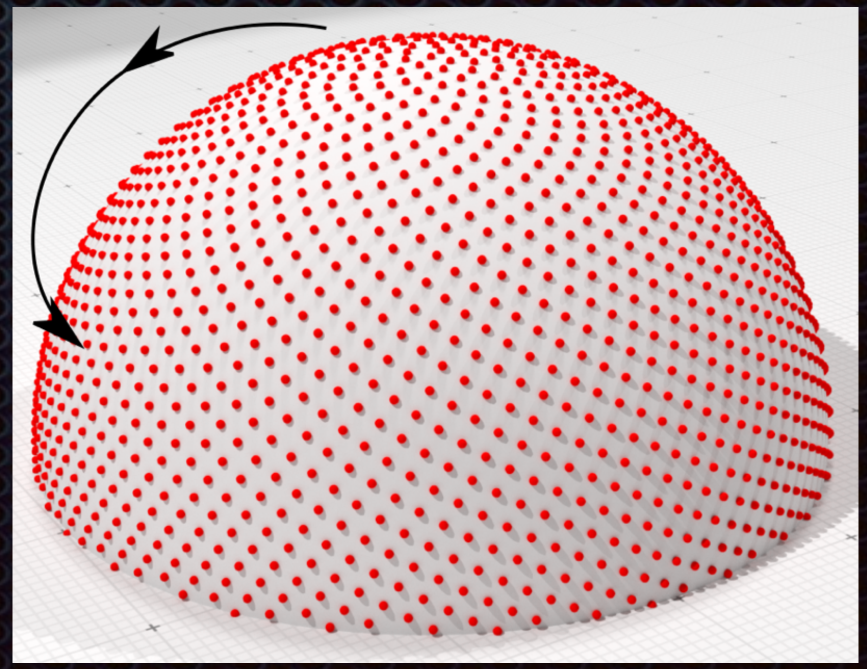


No rotation

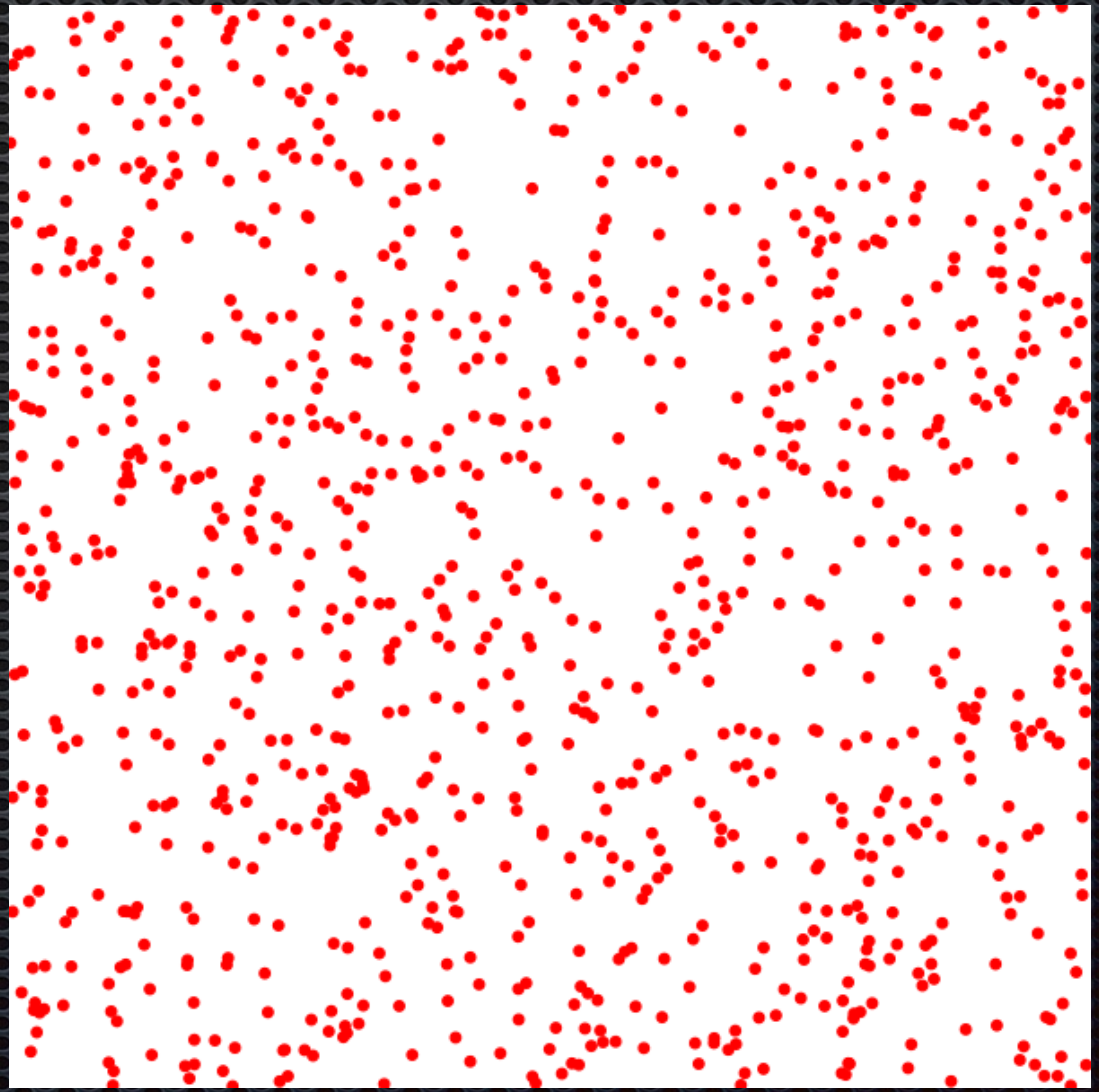


With rotation





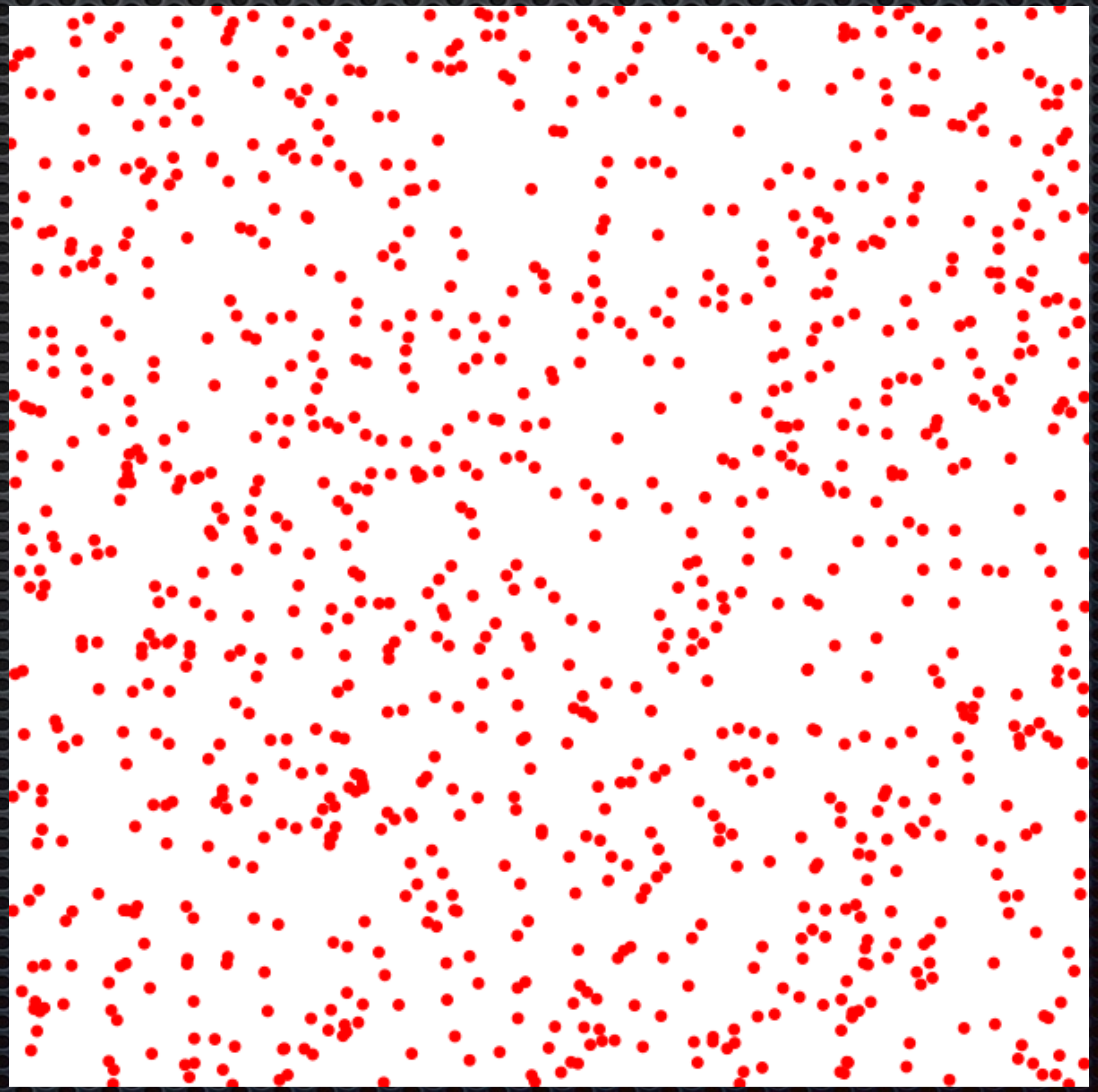
Homogeneous Sampling Pattern



Purely random samples

Homogeneous Sampling Pattern

Statistically invariant properties
over the domain

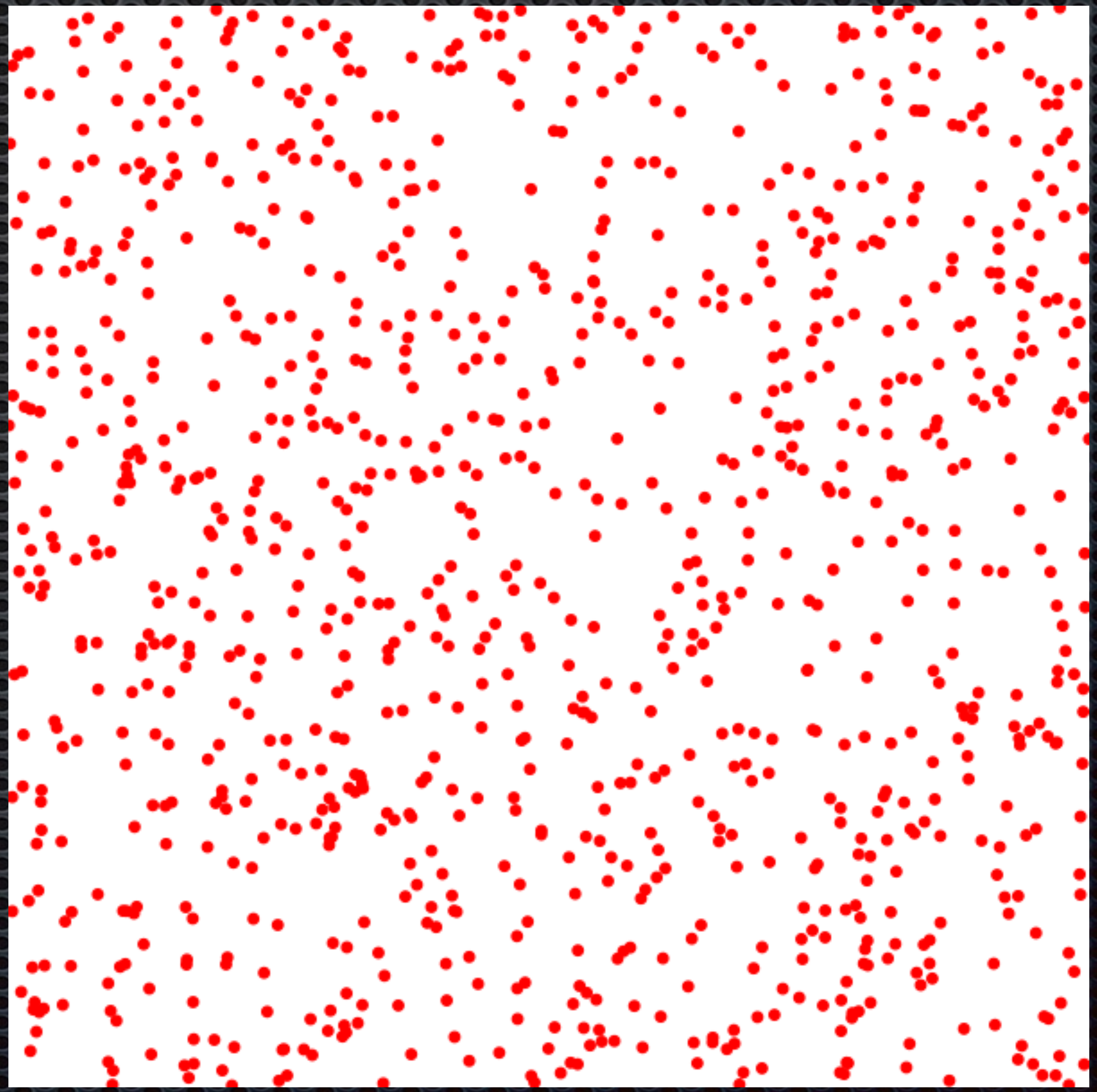


Purely random samples

Homogeneous Sampling Pattern

Statistically invariant properties over the domain

Widesense stationary
[Dippe and Wold 1985]



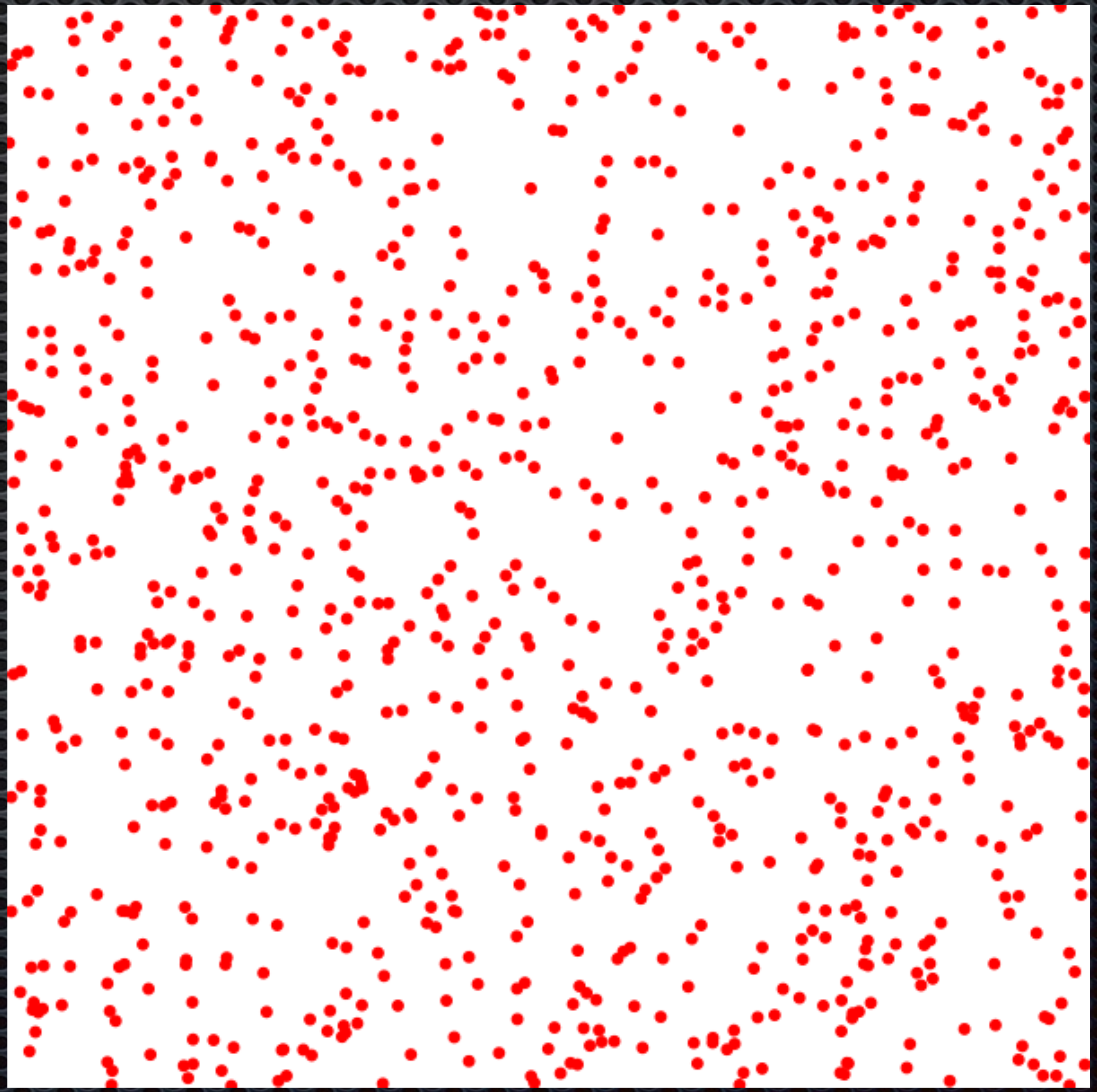
Purely random samples

Homogeneous Sampling Pattern

Statistically invariant properties over the domain

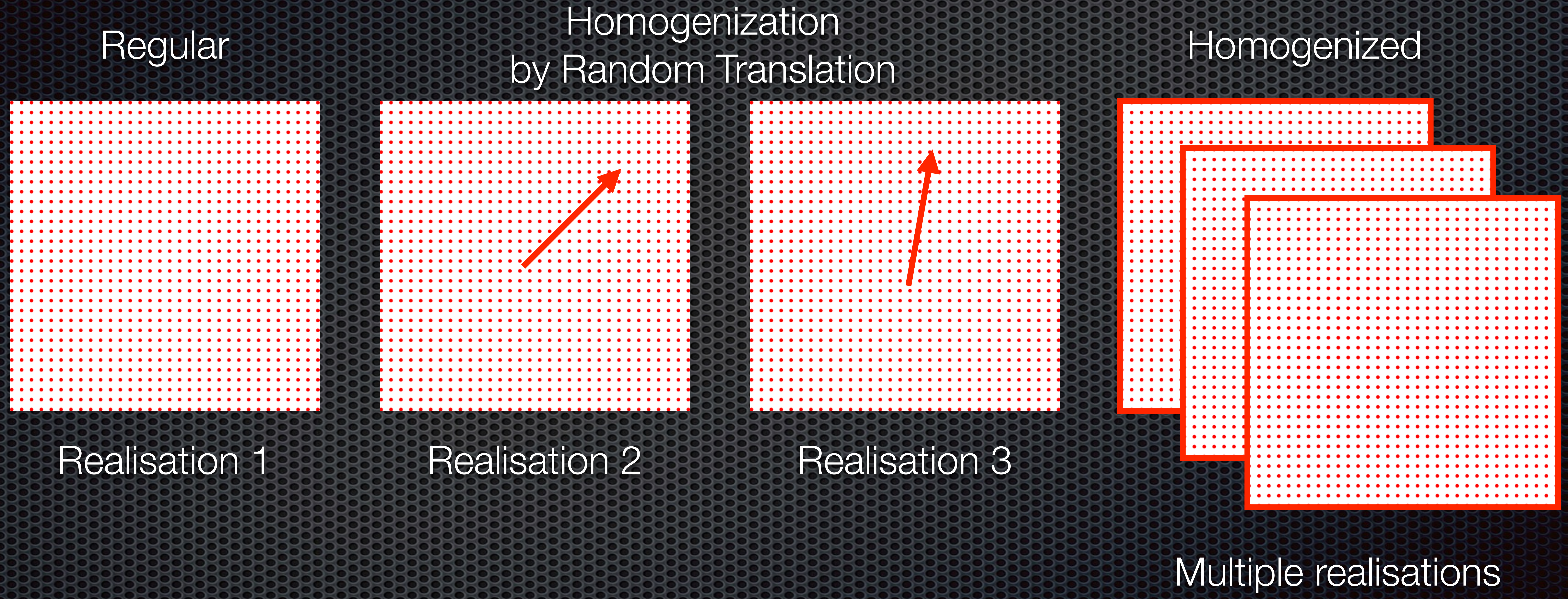
Widesense stationary
[Dippe and Wold 1985]

All sampling patterns derived from white noise

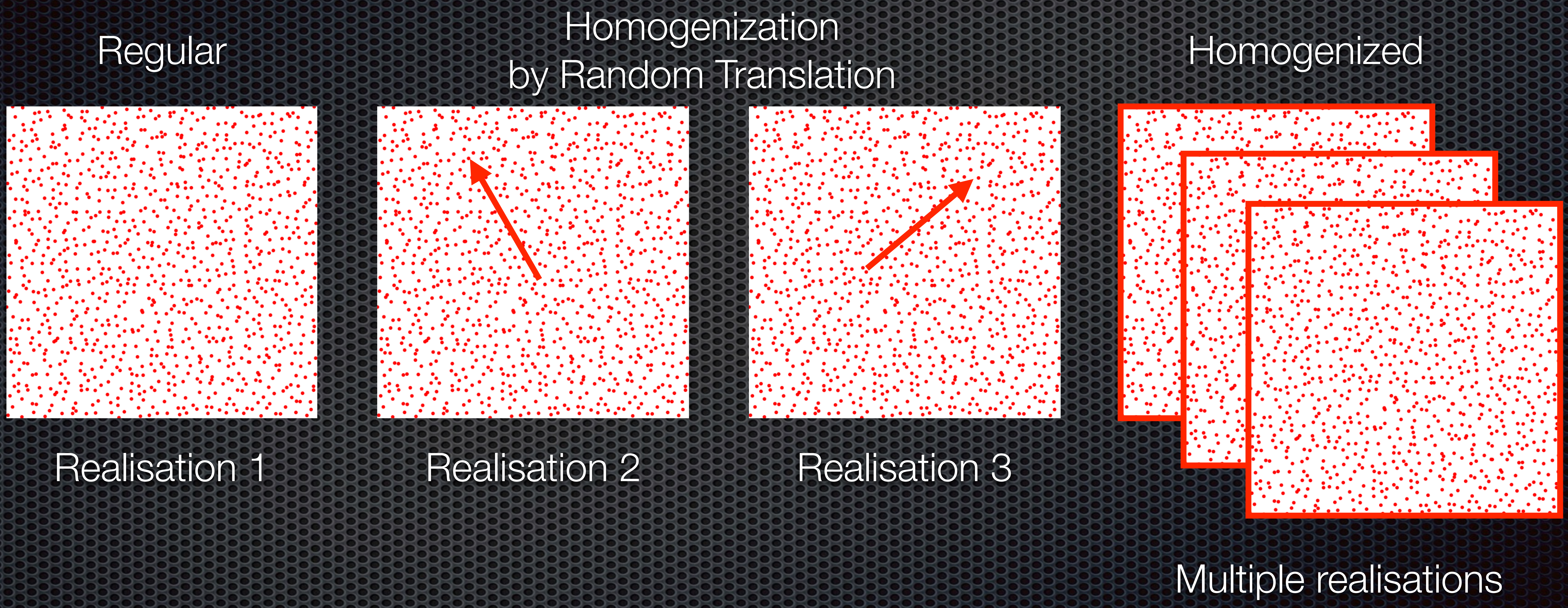


Purely random samples

Regular Samples



Jittered Samples

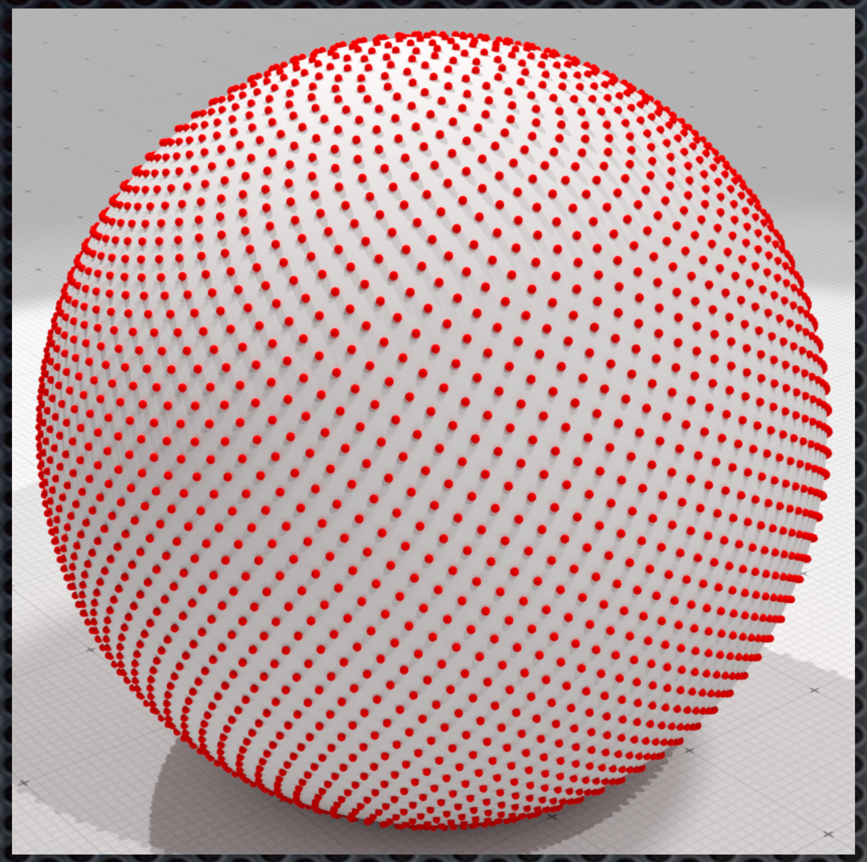


Homogeneous Samples

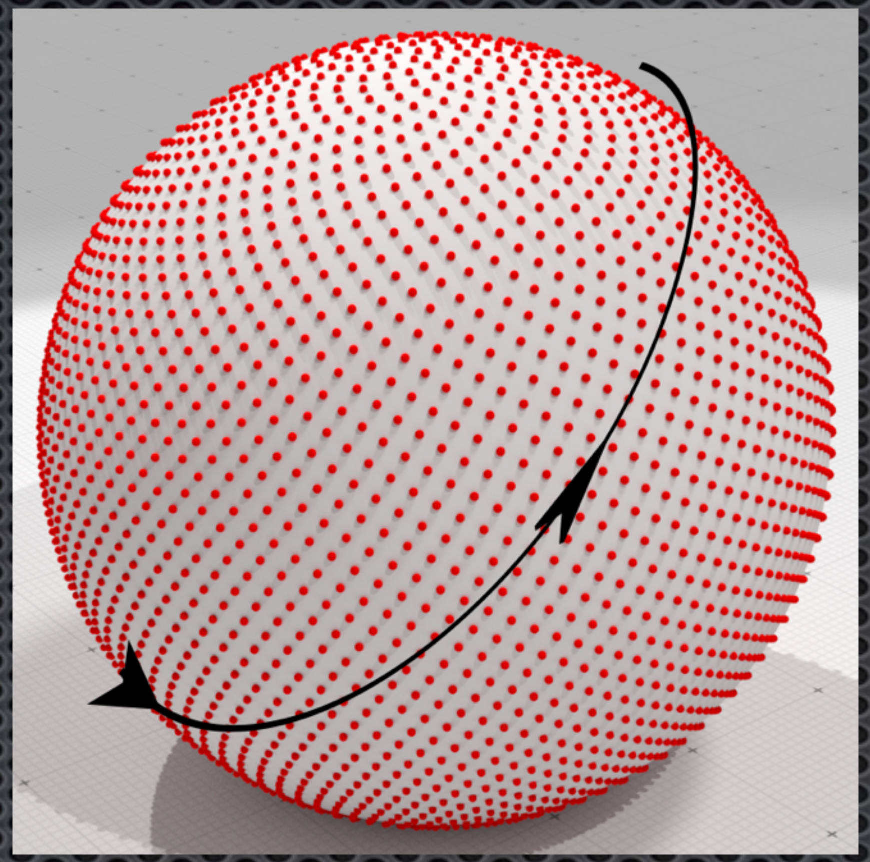
Regular

Homogenization
by Random Rotation

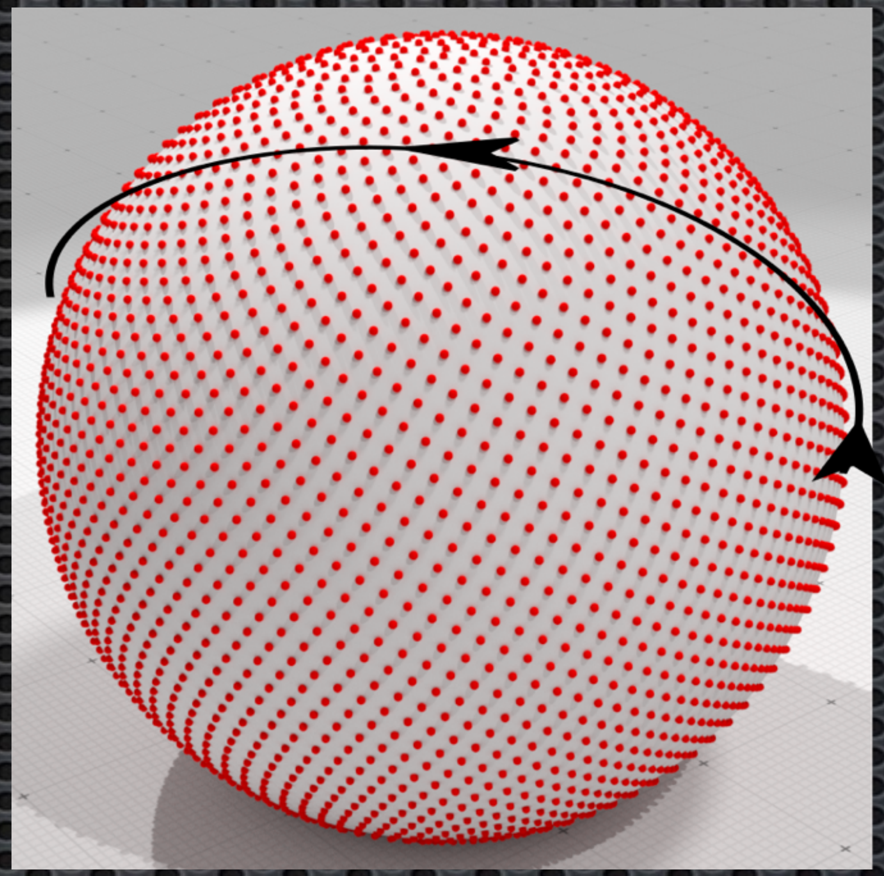
Homogenized



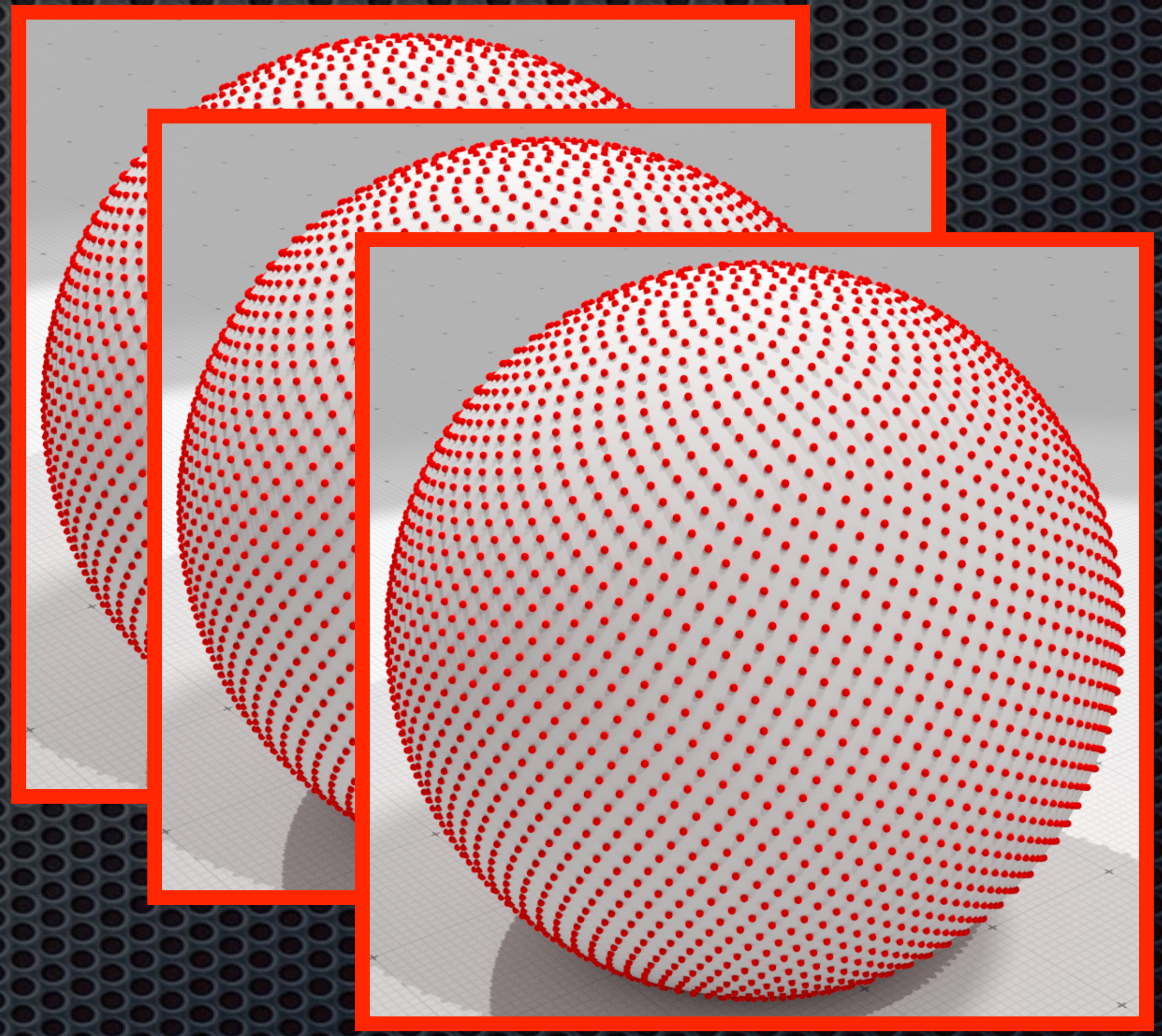
Realisation 1



Realisation 2



Realisation 3



Multiple realisations

Error in Terms of Variance

$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

Error in Terms of Variance

$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

Homogeneous Sampling:

Bias → Zero

Error in Terms of Variance

$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

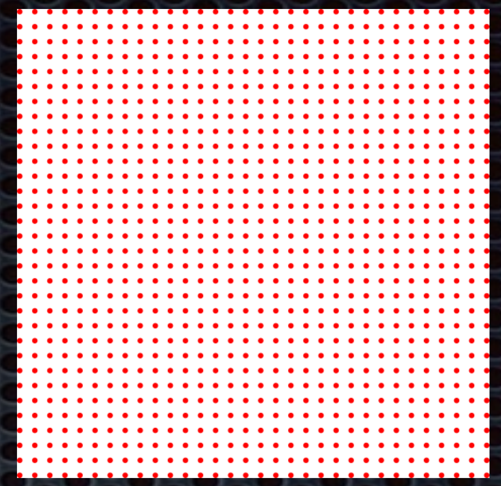
Homogeneous Sampling:

Bias → Zero

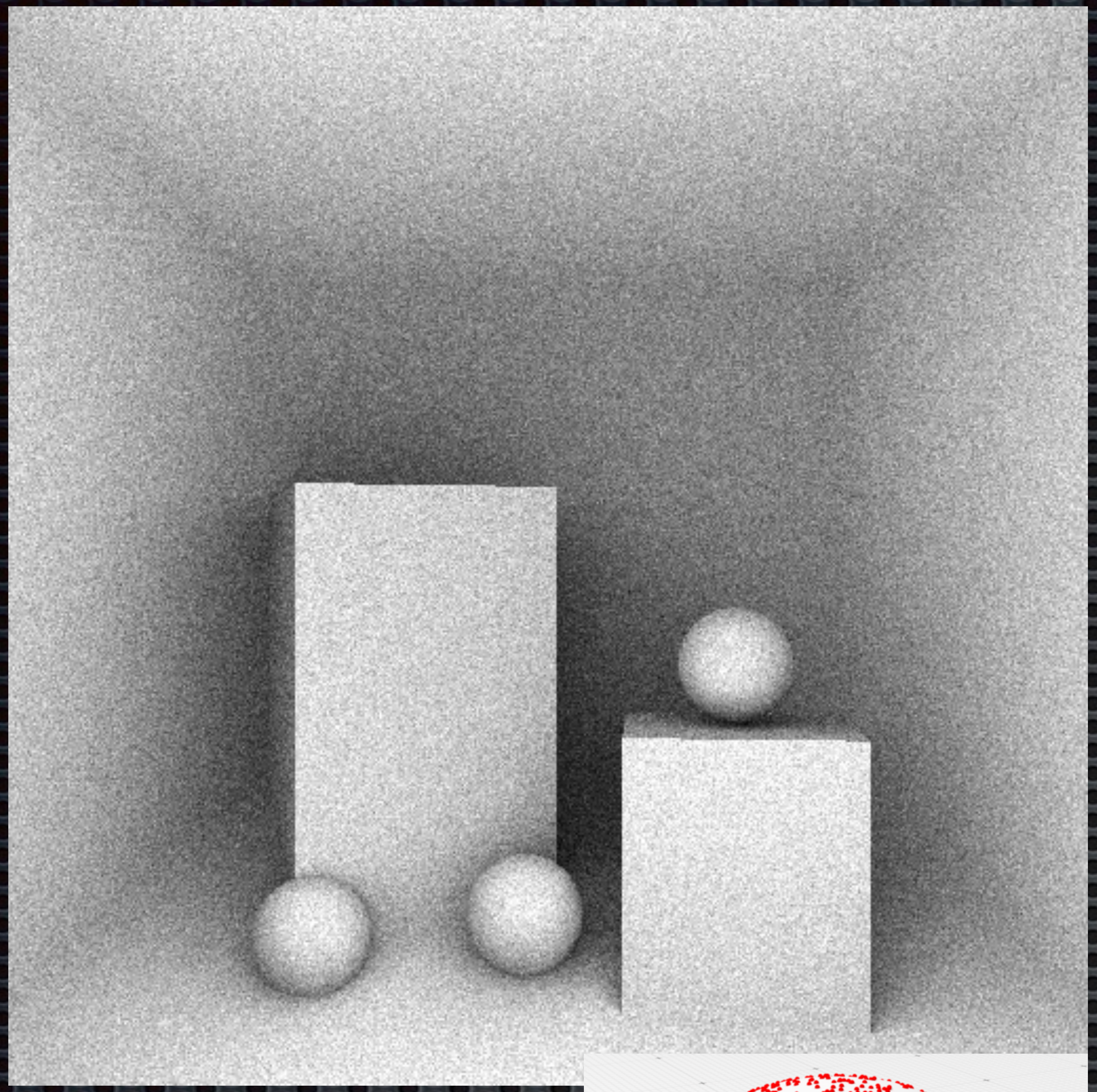
Implies:

$$\text{Error} = \text{Variance}$$

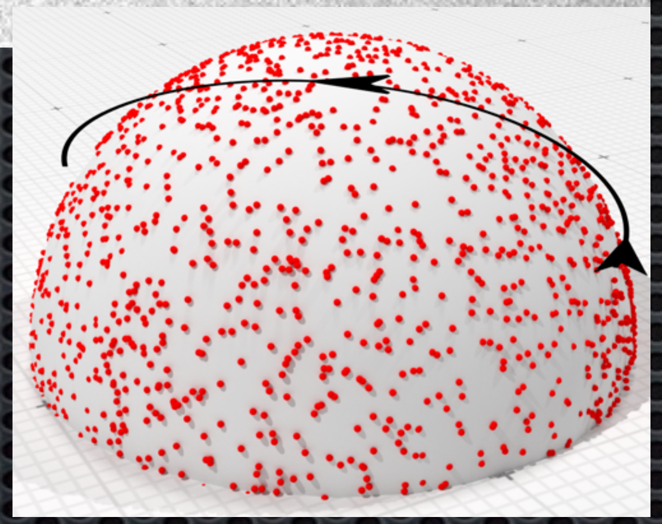
Nature of Noise



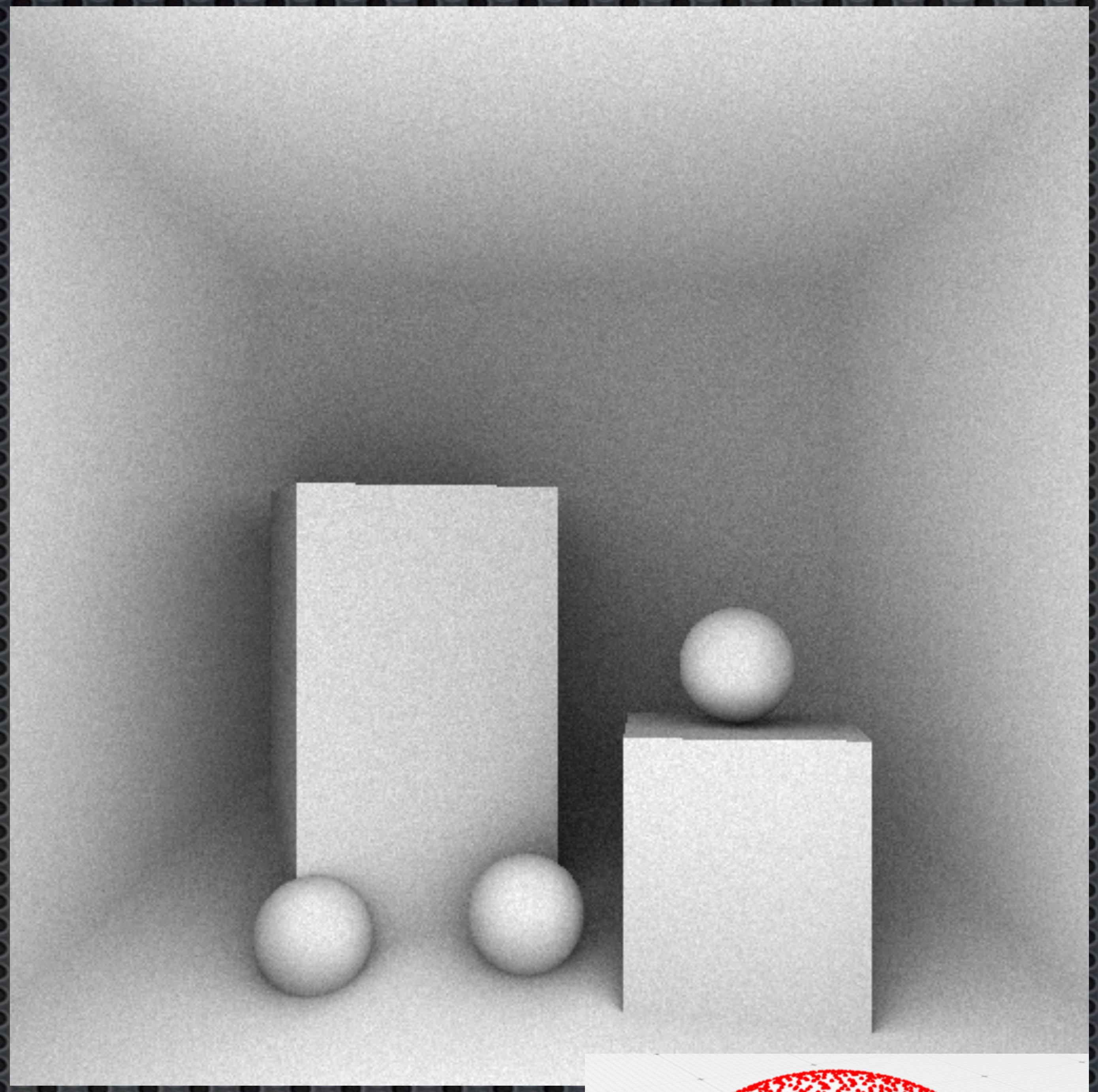
Purely Random



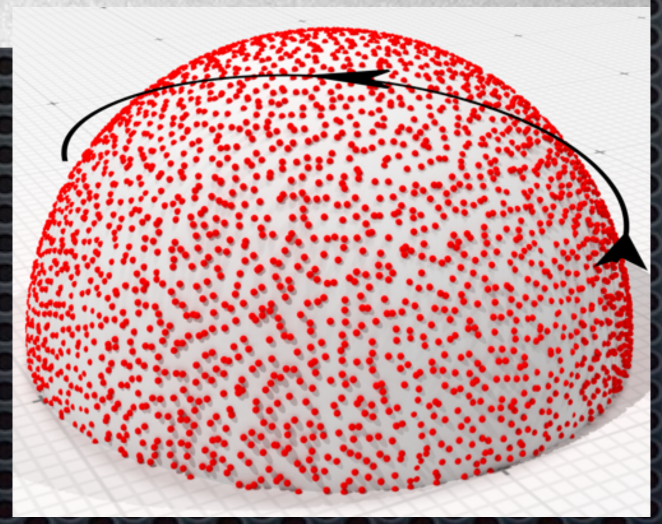
MSE:
4.79×10^{-3}



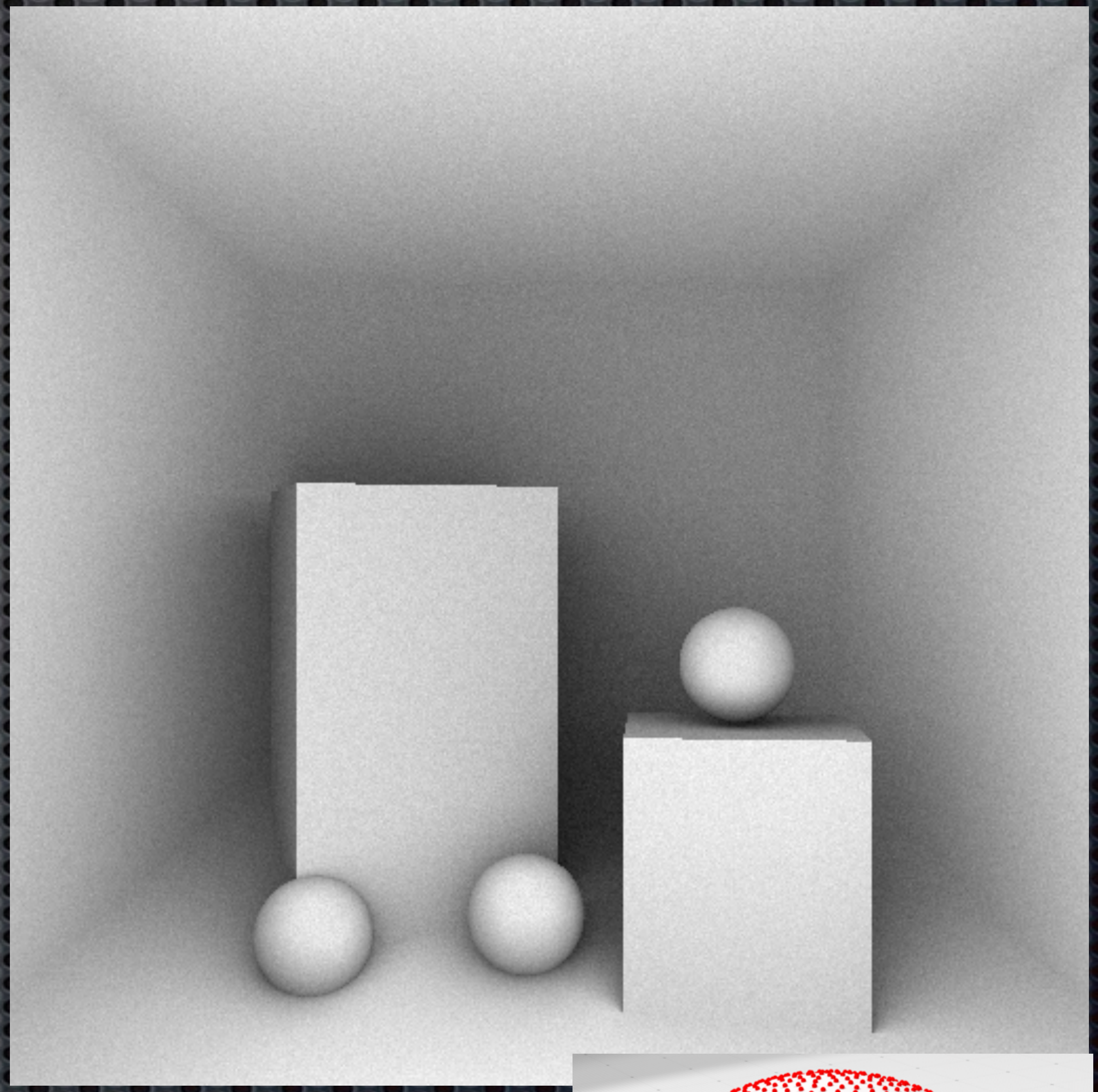
Jittered



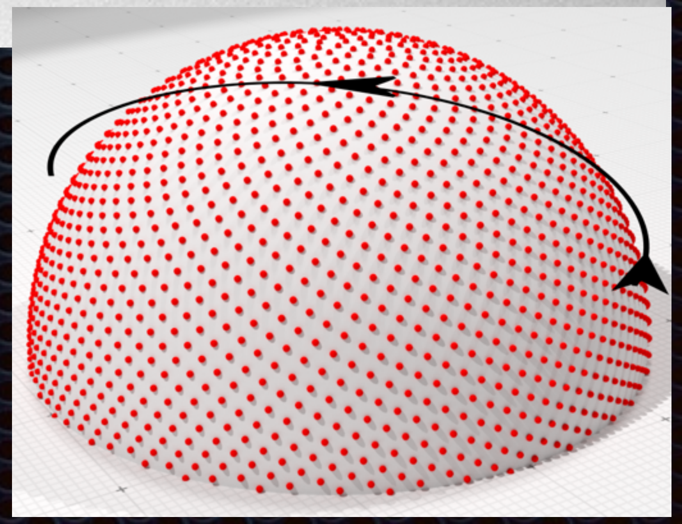
MSE:
8.56×10^{-4}



Regular



MSE:
3.95×10^{-4}



(96 hemispherical samples)

Variance
in Integration

?

Homogeneous
Sampling
Patterns

Variance in Integration

?

Homogeneous Sampling Patterns

How can we characterize sampling patterns ?

Previous Work on Fourier Analysis of Sampling Patterns

- Many prior works *[Dippé and Wold 1985], [Cook 1986], [Ulichney 1987]*

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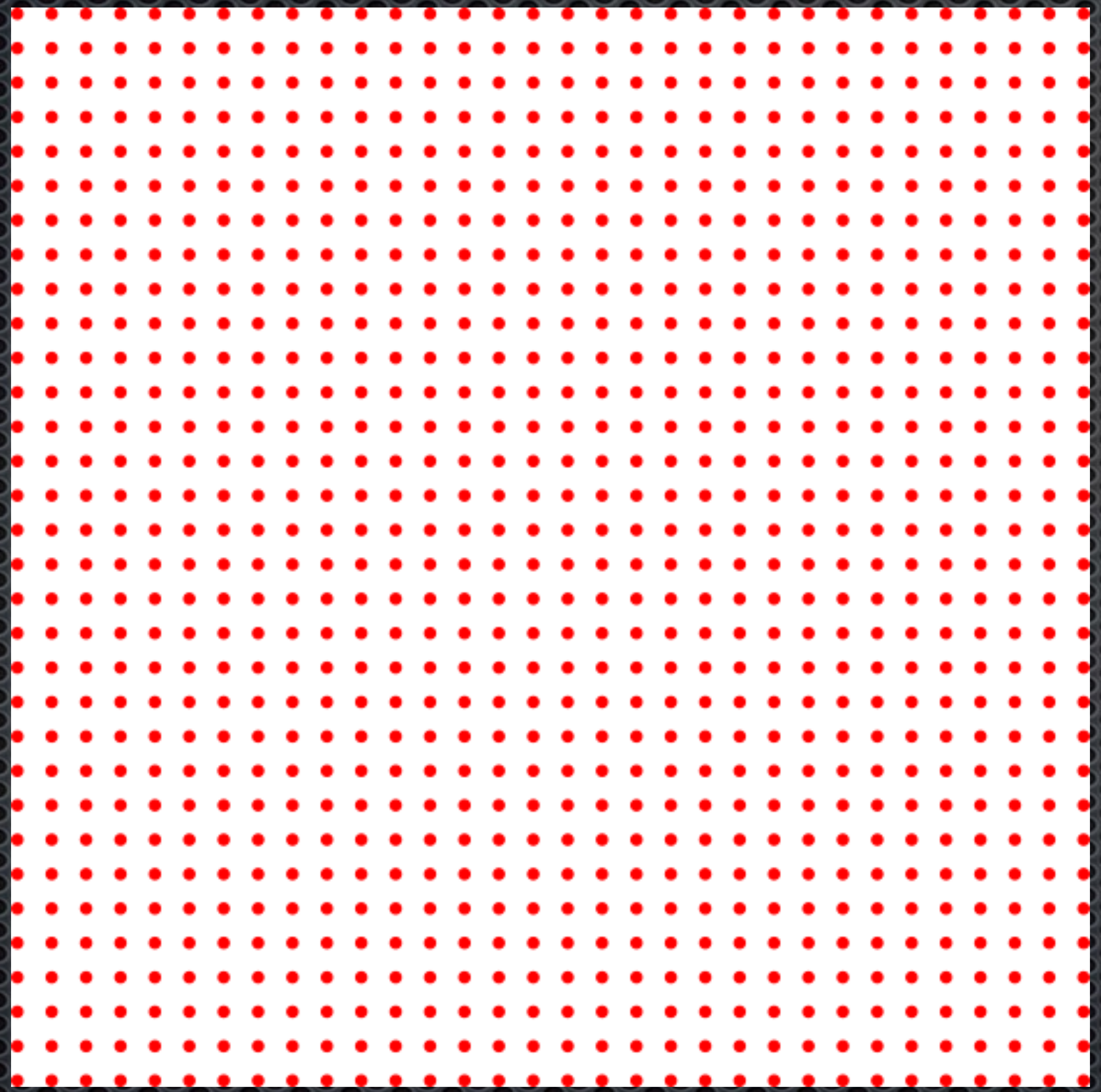
- Many prior works *[Dippé and Wold 1985], [Cook 1986], [Ulichney 1987]*
- Error relates to the frequency content of samples, *[Durand 2011]*

Previous Work on Fourier Analysis of Sampling Patterns

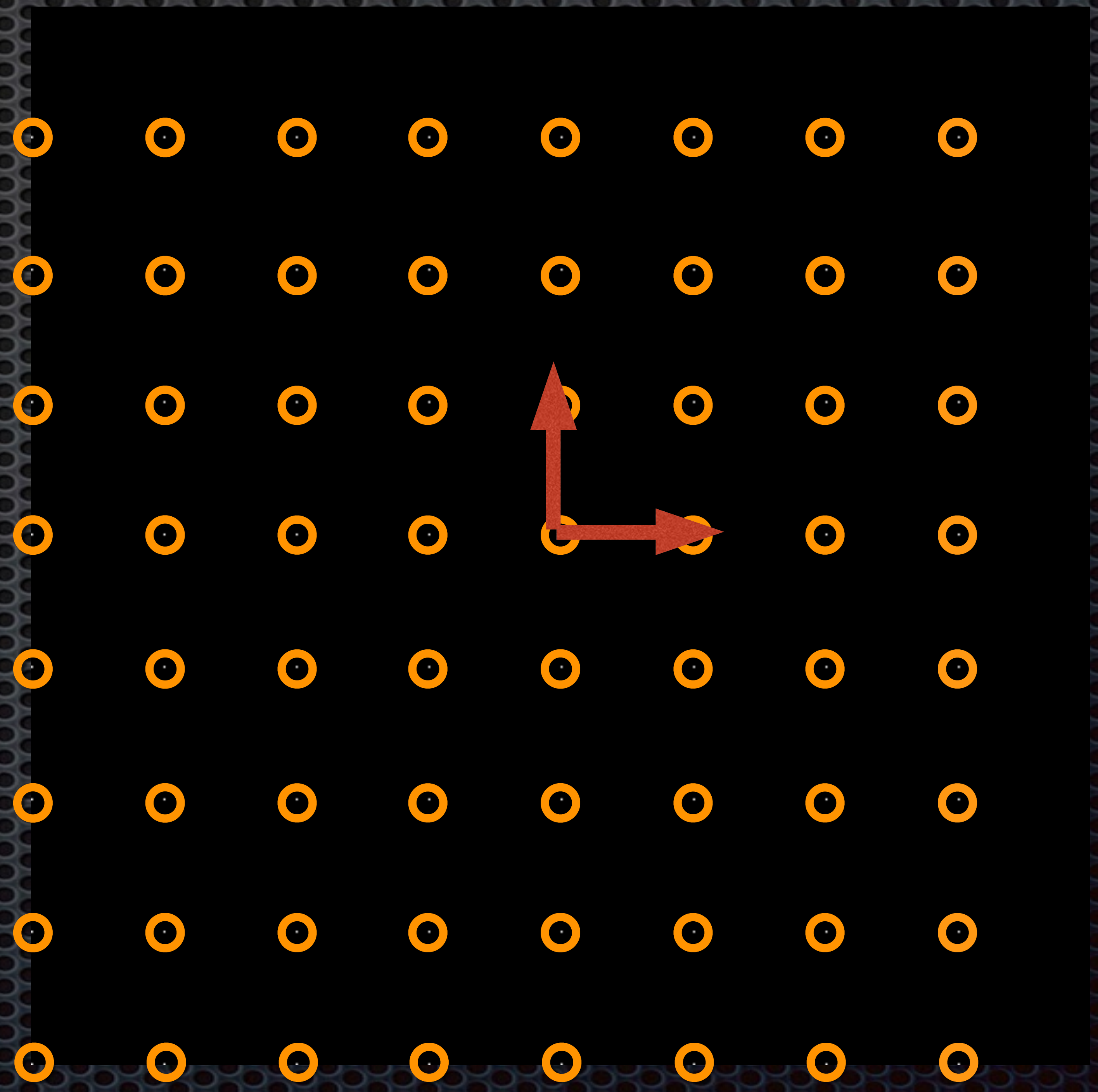
- Many prior works *[Dippé and Wold 1985], [Cook 1986], [Ulichney 1987]*
- Error relates to the frequency content of samples, *[Durand 2011]*
- Relates variance directly to the variance of Samples' Fourier Coefficients *[Subr and Kautz 2013]*

Regular Sampling Pattern

Samples

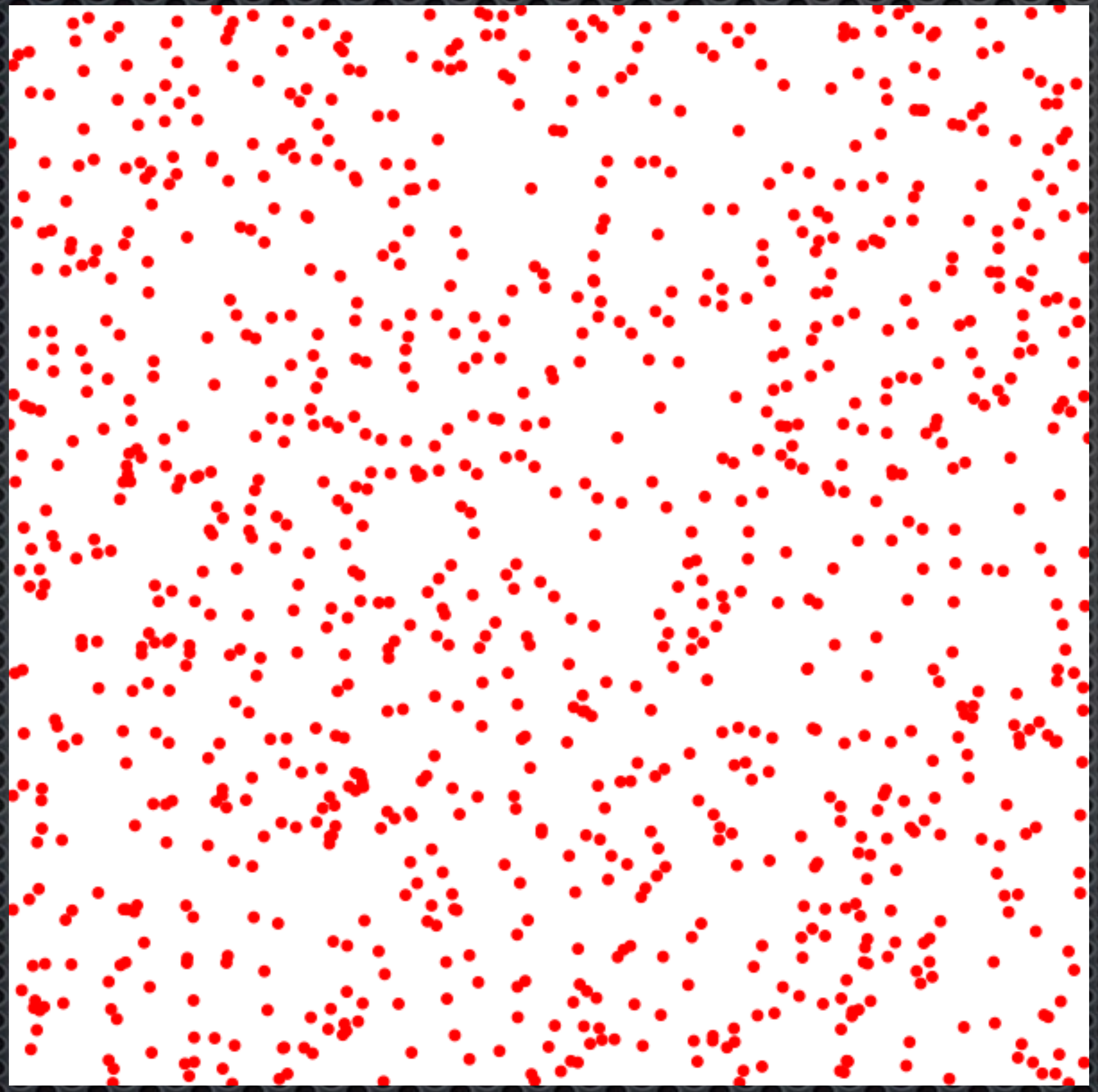


Power Spectrum

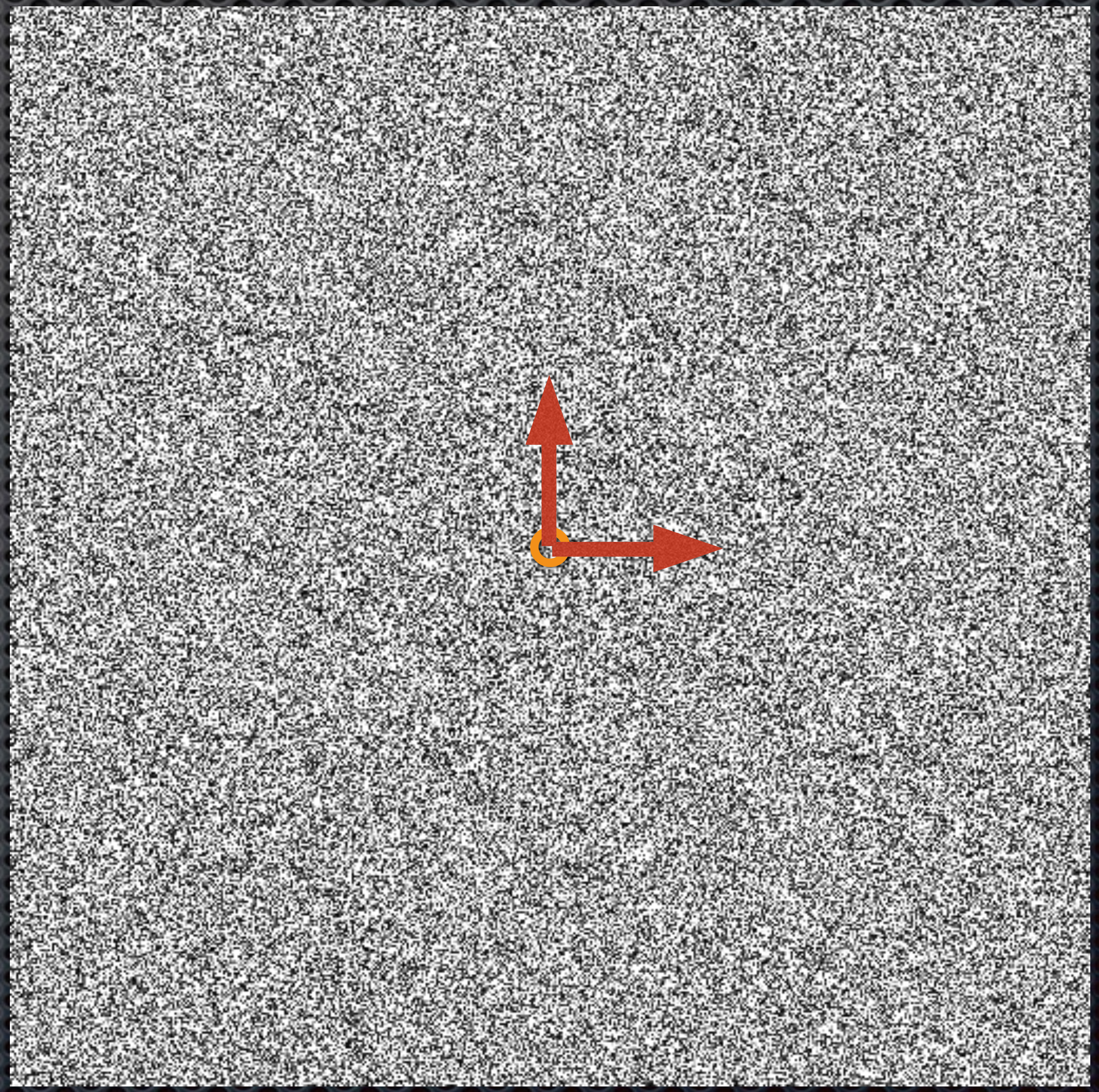


Purely Random Sampling Pattern

Samples

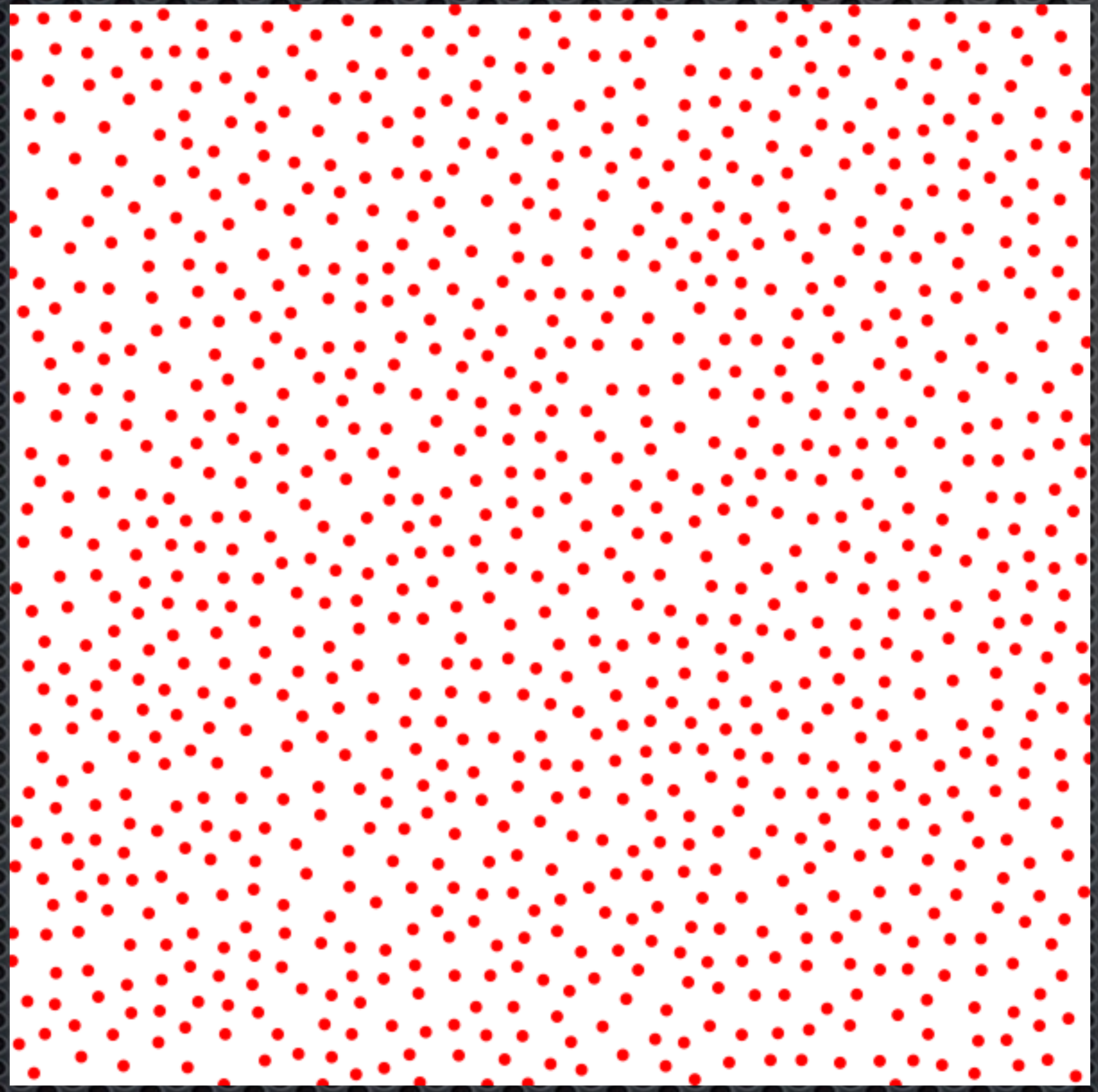


Power Spectrum

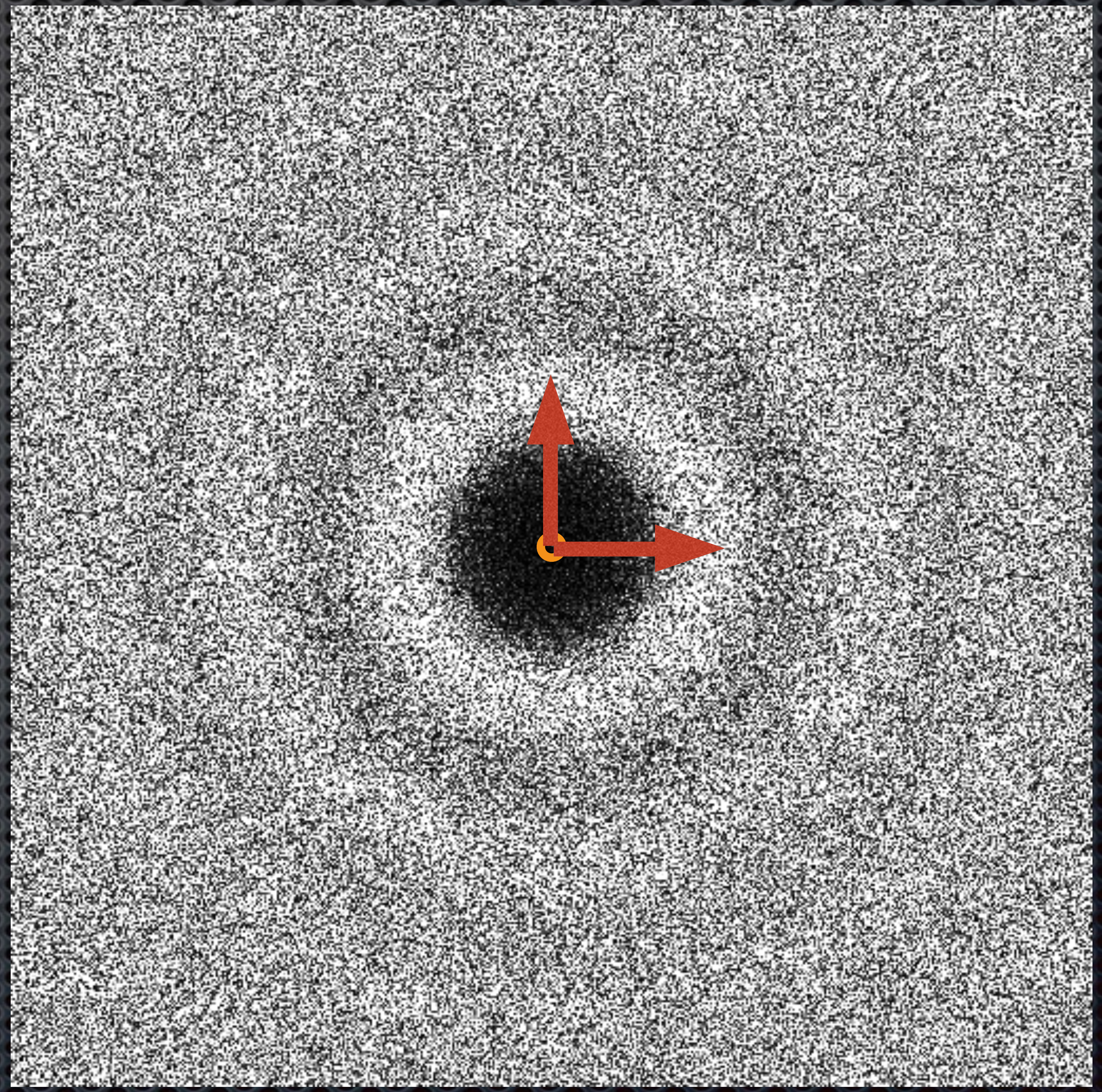


Poisson Disk Sampling Pattern

Samples

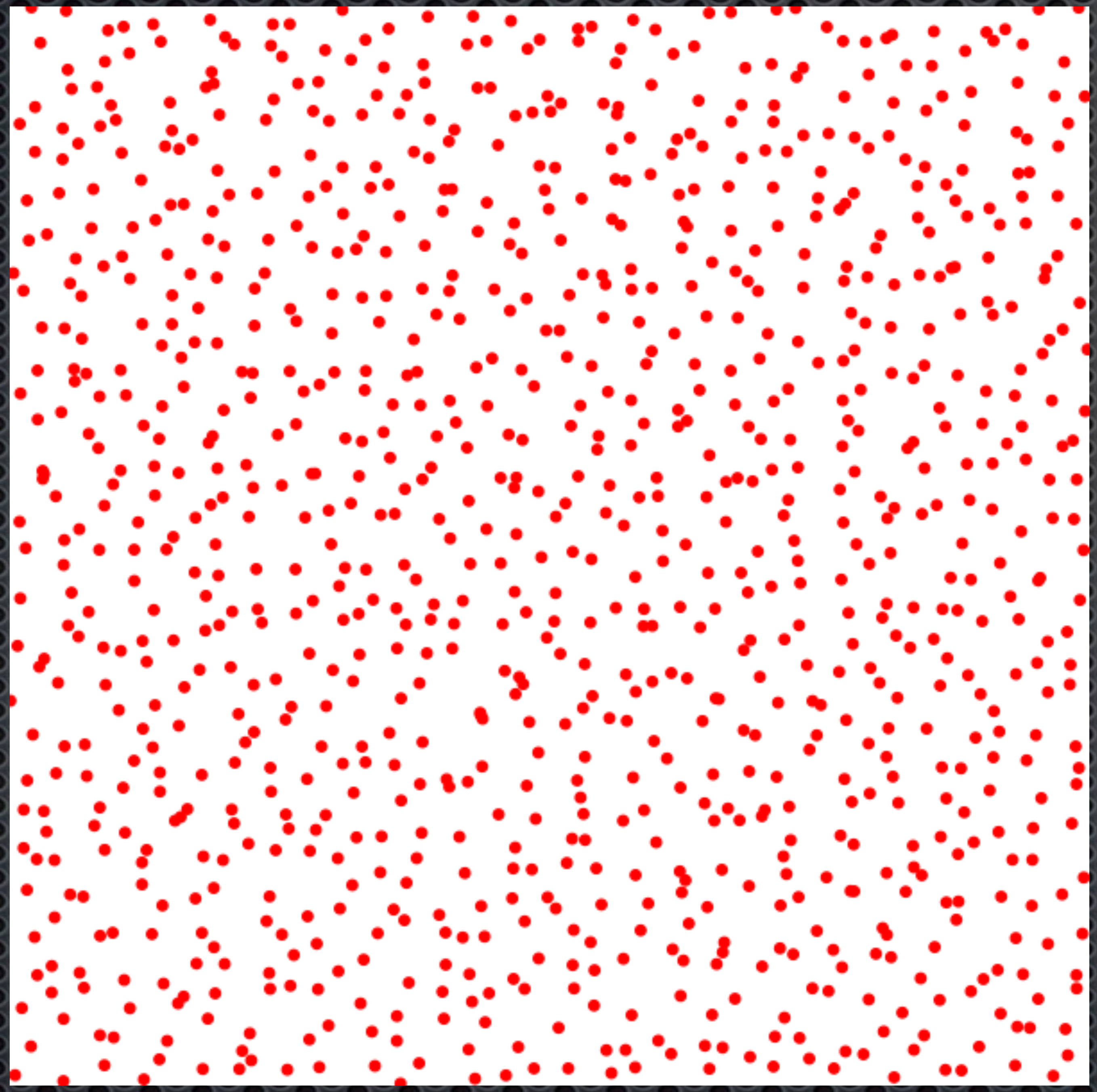


Power Spectrum

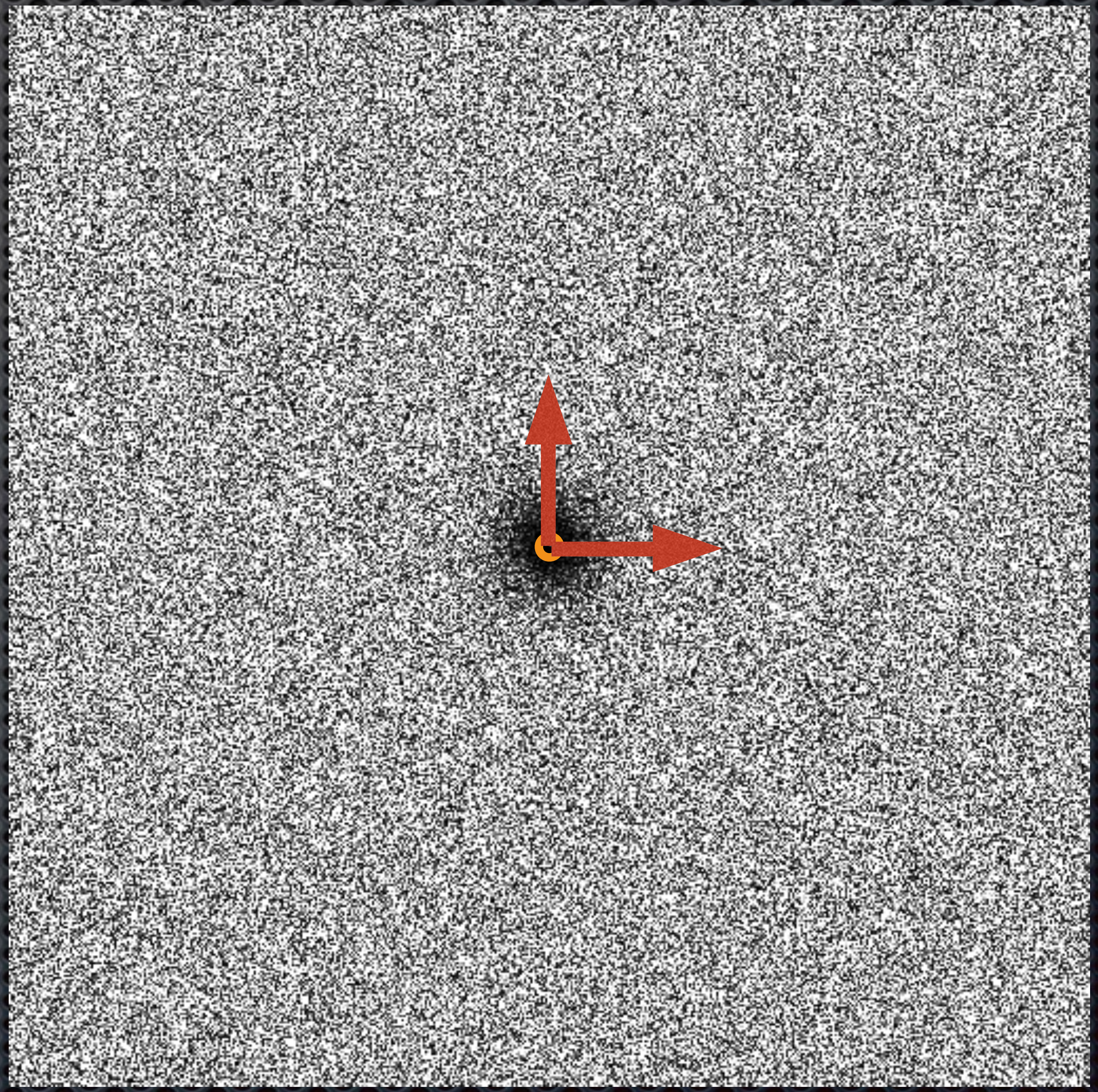


Jittered Sampling Pattern

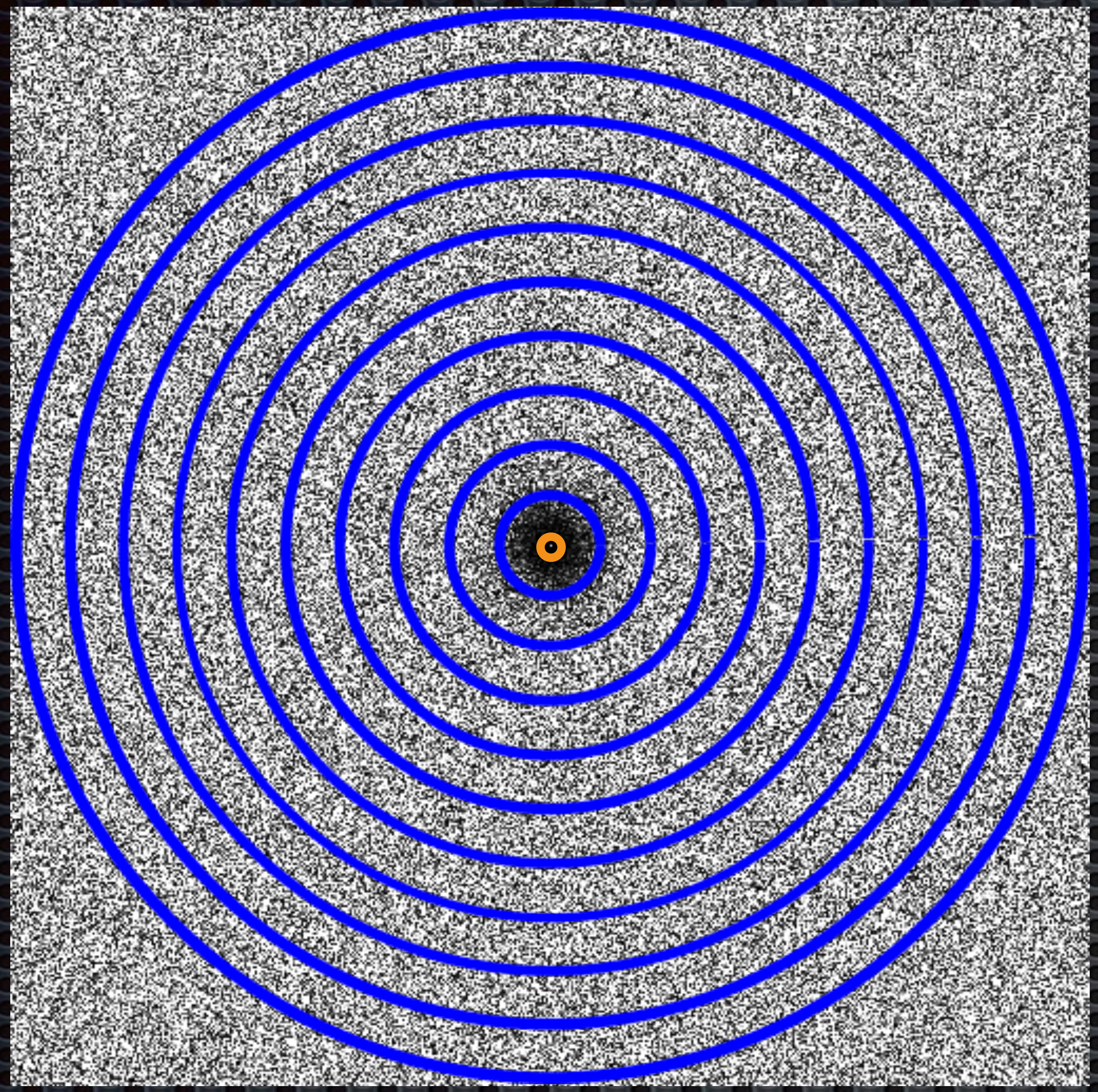
Samples

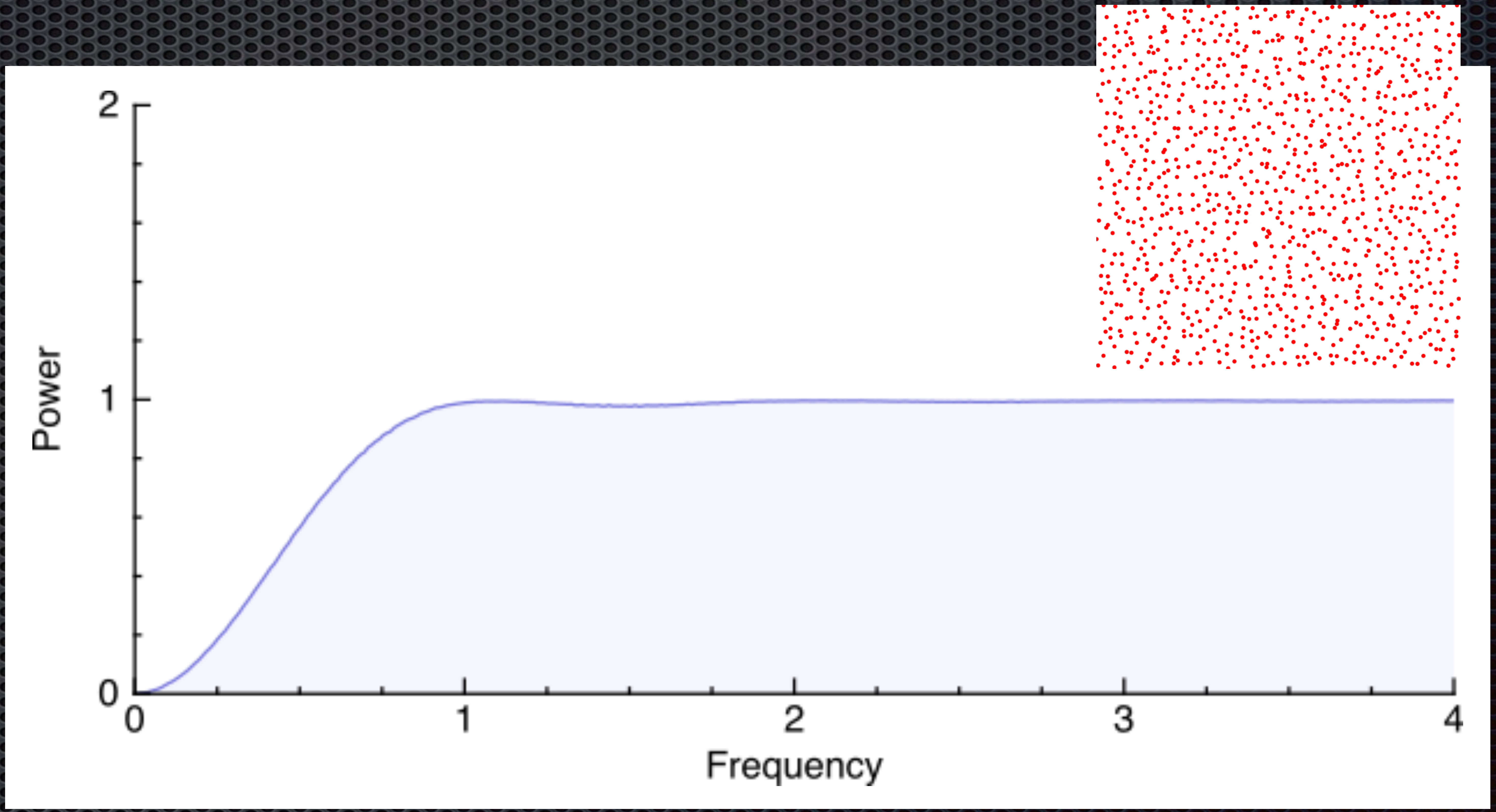


Power Spectrum



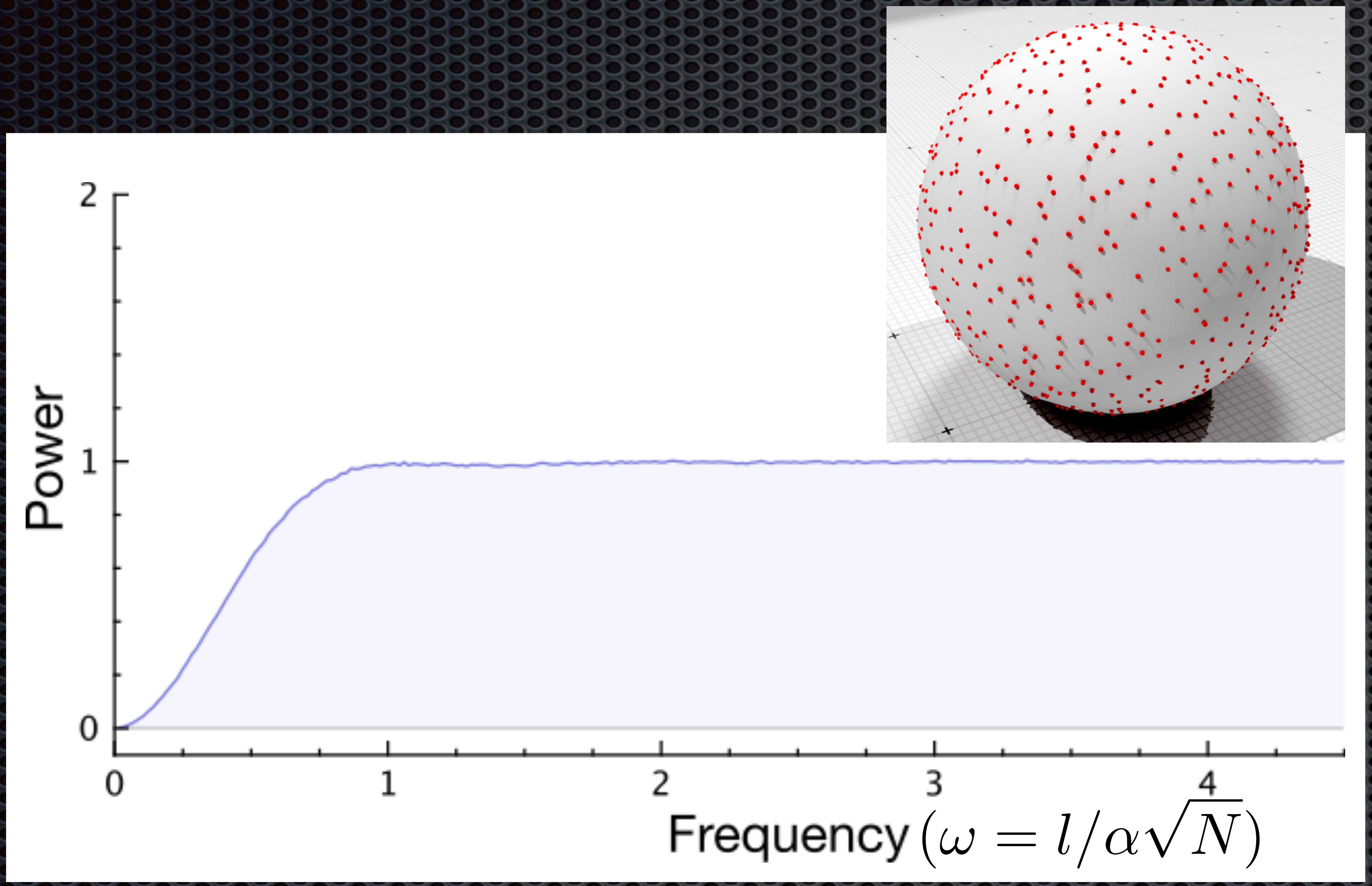
Radial Averging of Power Spectrum

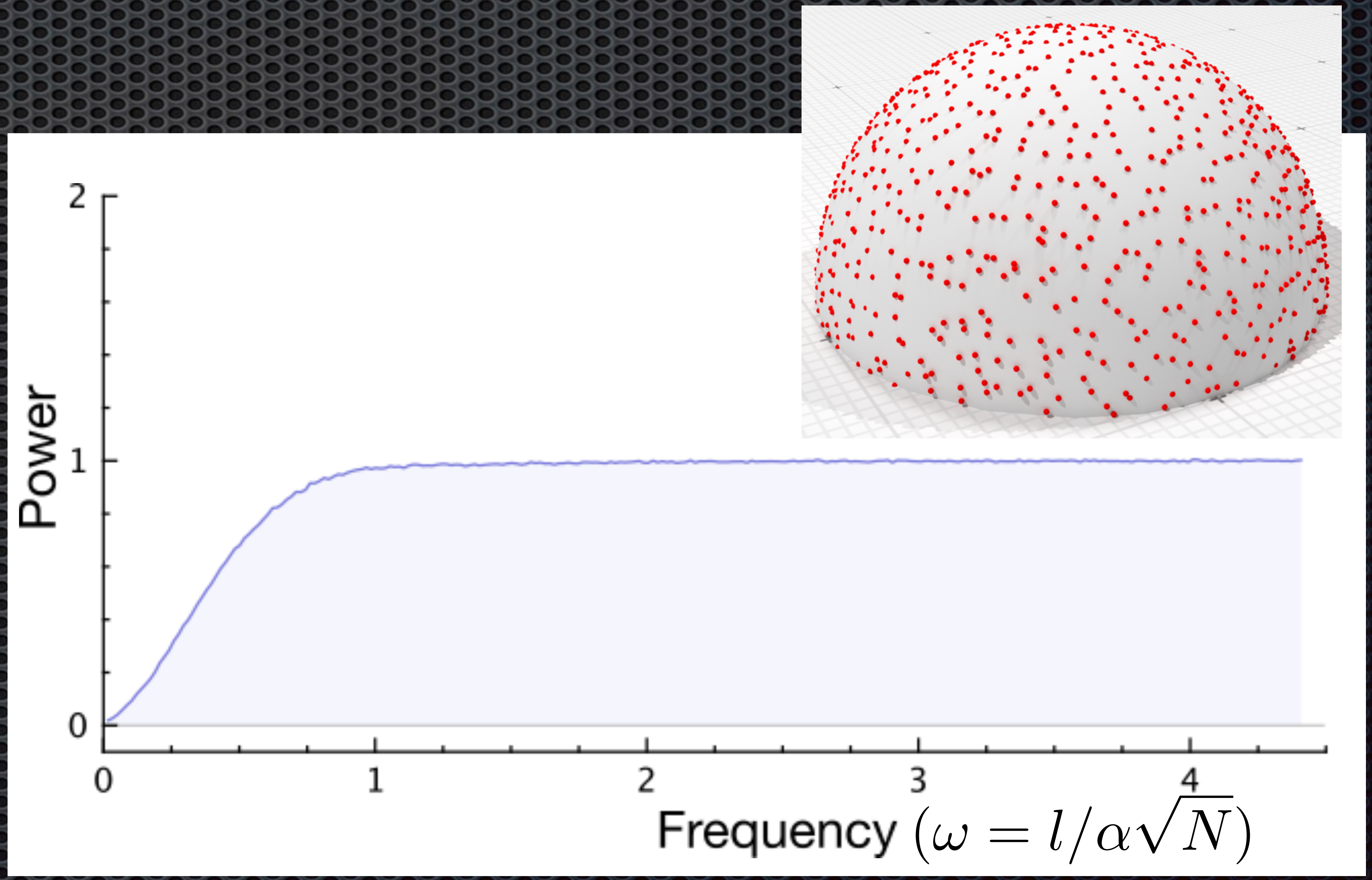




(Jittered Sampling Pattern)

Mean Angular Power Spectrum





Jittered Sampling Pattern

Variance
in Integration

?

Homogeneous
Samples
+
Frequency content
(Power Spectra)

Variance Formulation in Euclidean Domain

$$Var(\boldsymbol{I}_N) = \frac{\mu(\mathcal{T}^d)\mu(S^{d-1})}{N} \int_0^\infty \rho^{d-1} \breve{\mathcal{P}}_{\mathbf{S}}(\rho) \breve{\mathcal{P}}_{\mathbf{F}}(\rho) \mathsf{d}\rho$$

Variance Formulation in Euclidean Domain

$$Var(\boldsymbol{I}_N) = \frac{\mu(\mathcal{T}^d)\mu(S^{d-1})}{N} \int_0^\infty \rho^{d-1} \breve{\mathcal{P}}_{\mathbf{S}}(\rho) \breve{\mathcal{P}}_{\mathbf{F}}(\rho) \mathsf{d}\rho$$

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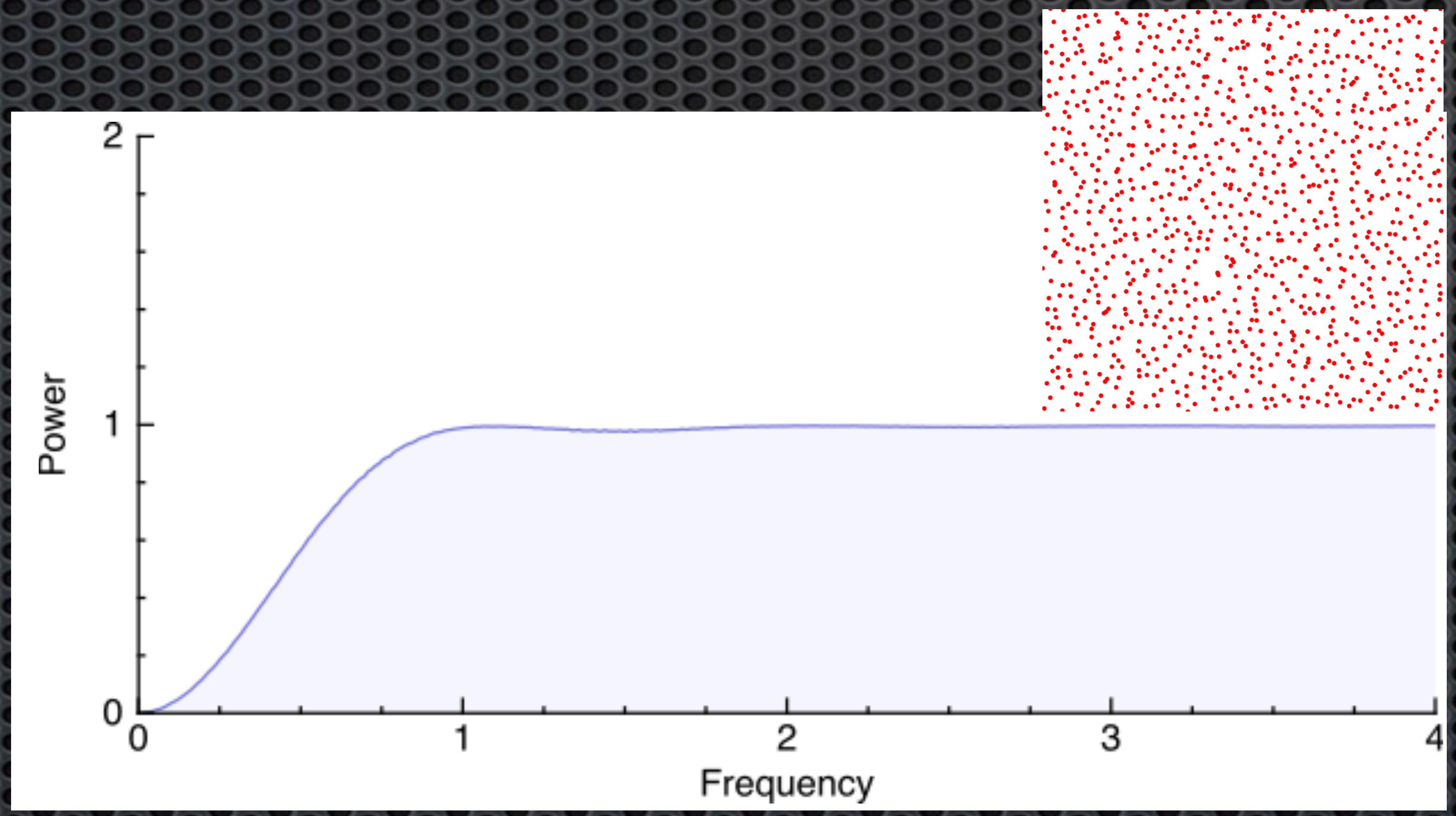
$$\breve{\mathcal{P}}_{\mathbf{S}}(\rho) =$$



(Jittered Sampling Pattern)

Variance Formulation in Spherical Domain

$$Var(\boldsymbol{I}_N) = \frac{\mu(S^2)}{N} \sum_{l=1}^{\infty} (2l+1) \breve{\mathcal{P}}_{\mathbf{S}}(l) \breve{\mathcal{P}}_{\mathbf{F}}(l)$$

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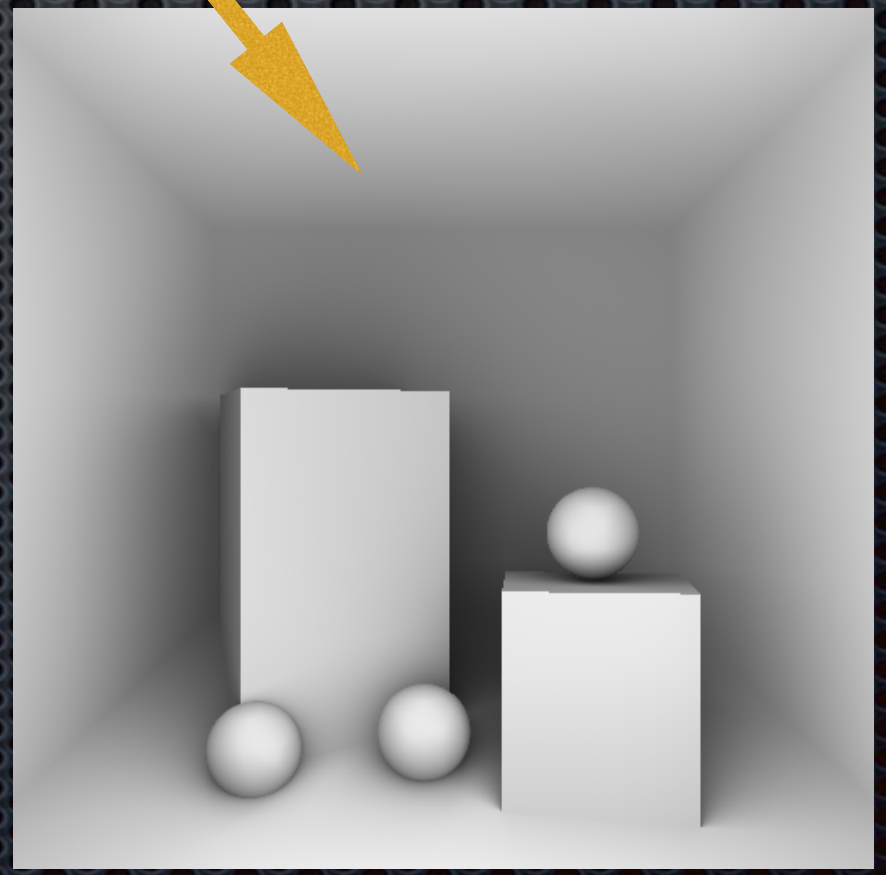

Variance Formulation in Spherical Domain

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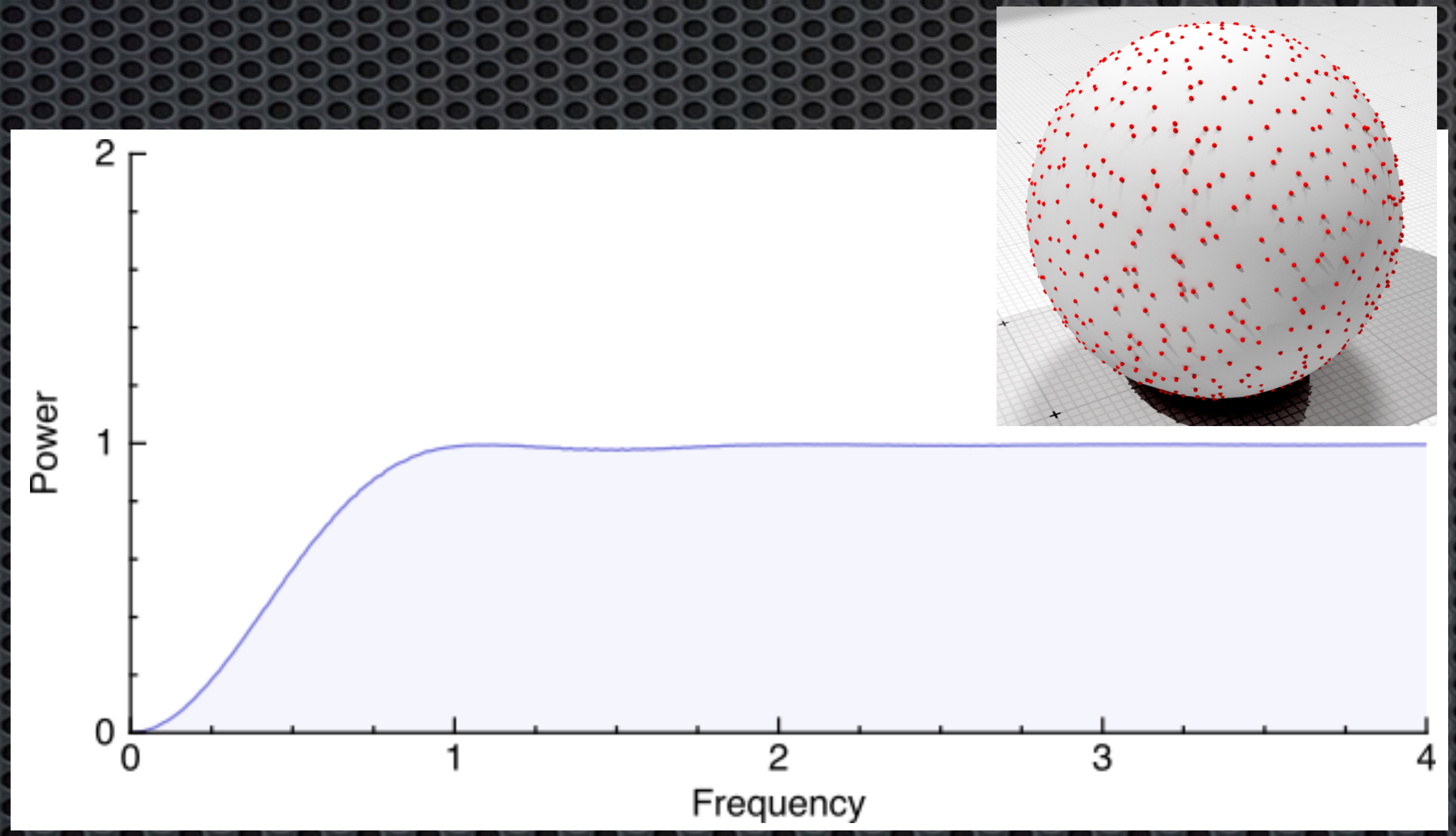
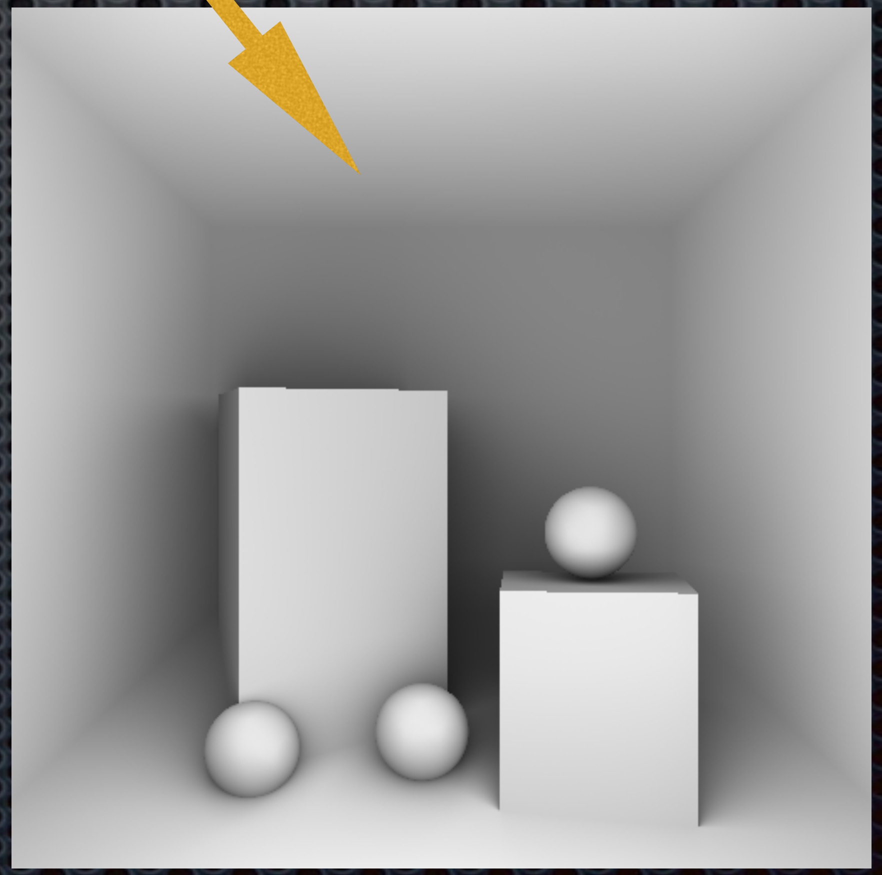
$$\breve{\mathcal{P}}_{\mathbf{S}}(\omega) =$$

$$\omega = l/\alpha\sqrt{N}$$



(Jittered Sampling Pattern)



Variance: Product of $\breve{\mathcal{P}}_{\mathbf{S}}(\cdot)$ and $\breve{\mathcal{P}}_{\mathbf{F}}(\cdot)$

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$$Var(\boldsymbol{I}_N) \propto \frac{1}{N} \int_0^\infty \rho^{d-1} \breve{\mathcal{P}}_{\mathbf{S}}(\rho) \breve{\mathcal{P}}_{\mathbf{F}}(\rho) \mathrm{d}\rho$$

Euclidean

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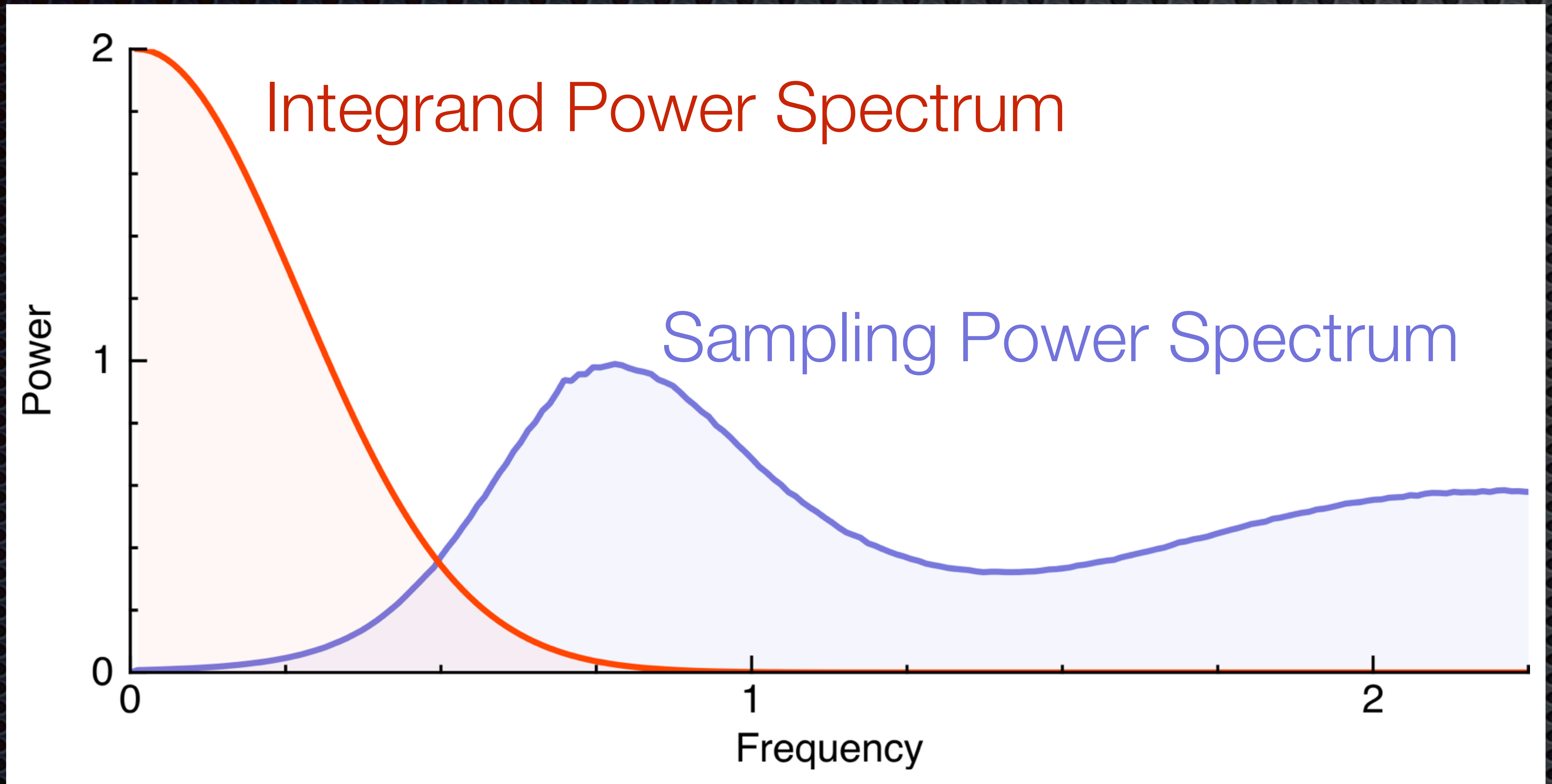
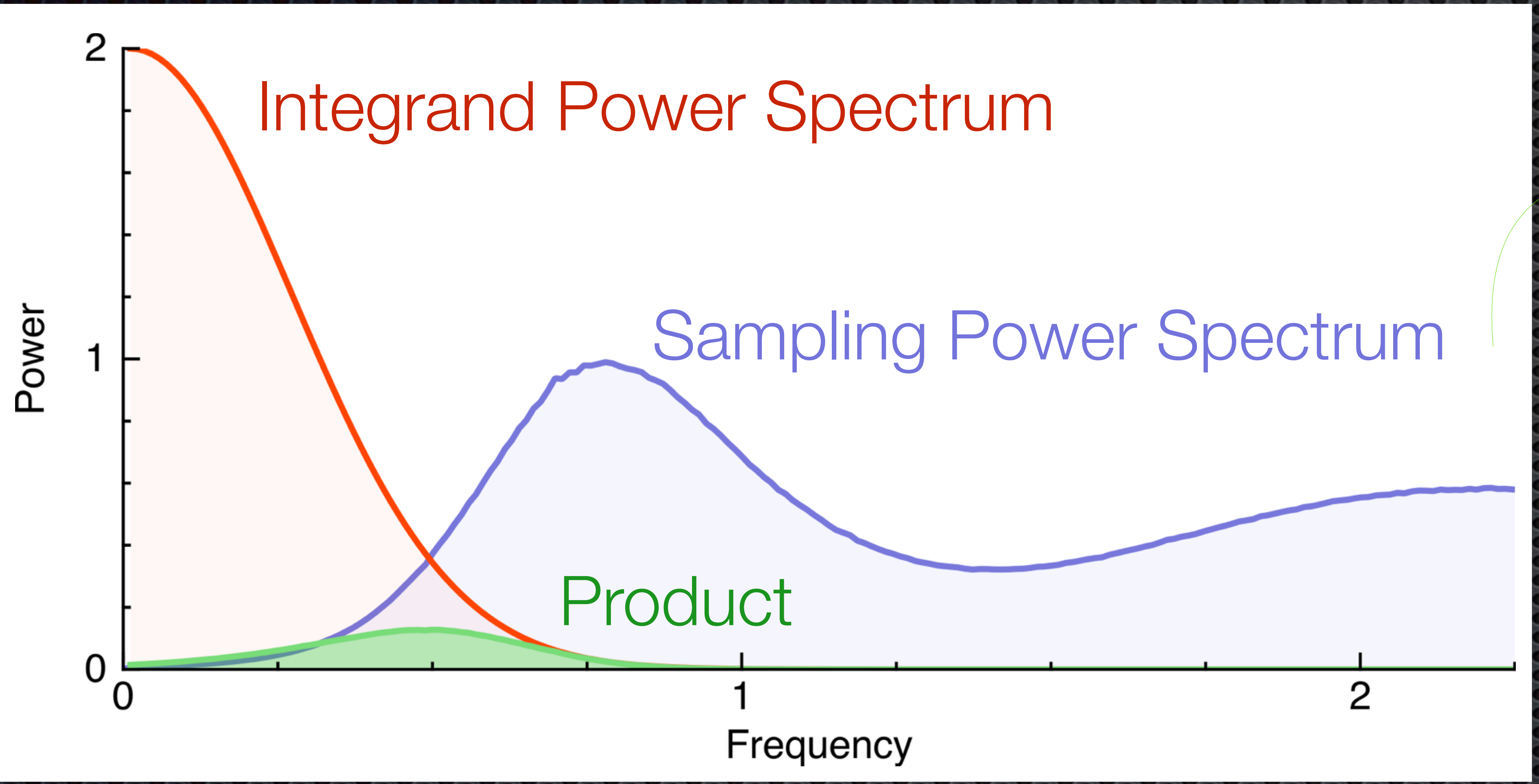
Spherical

Dependence on Number of Samples



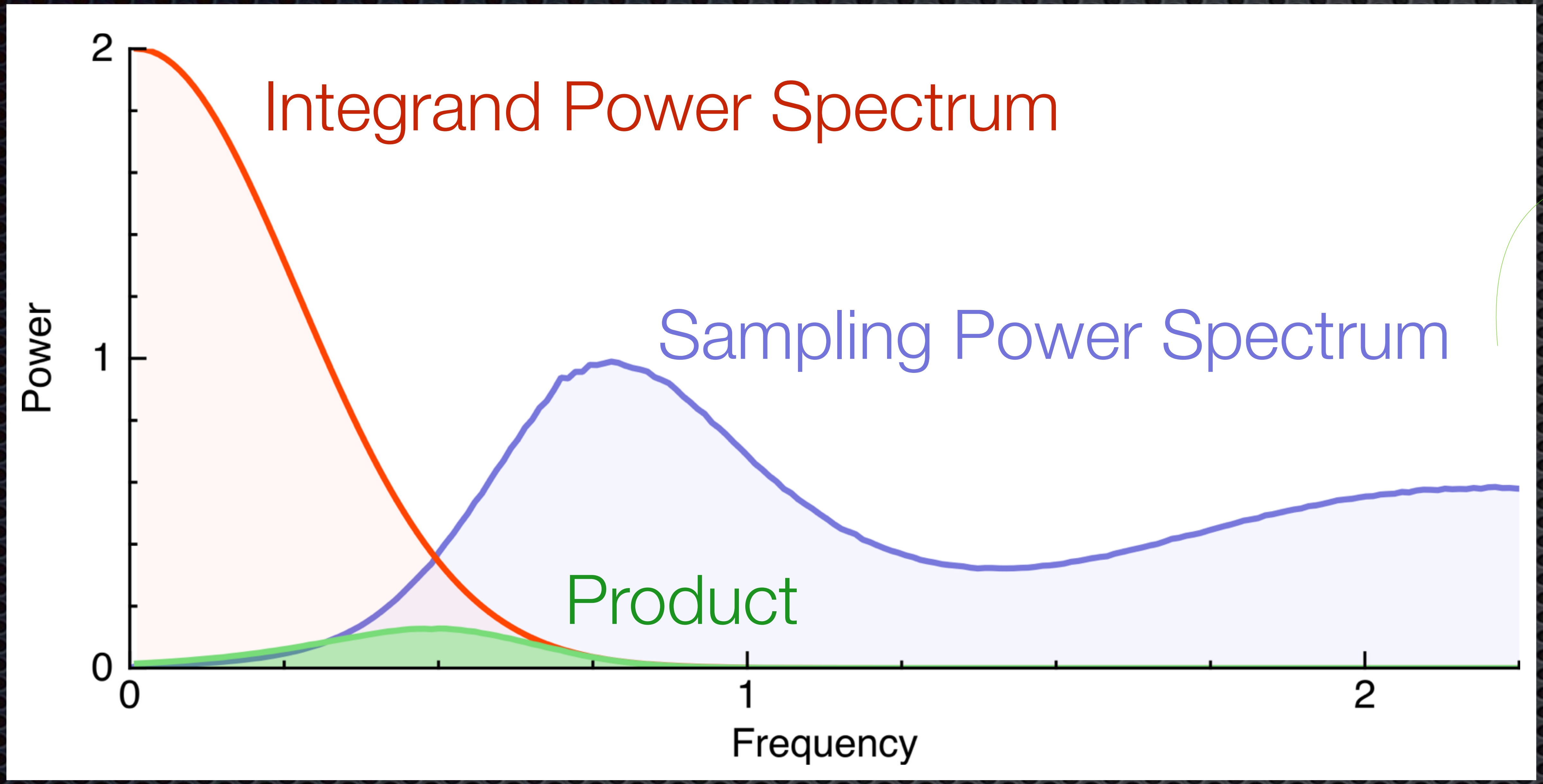
For a given number of samples

Dependence on Number of Samples

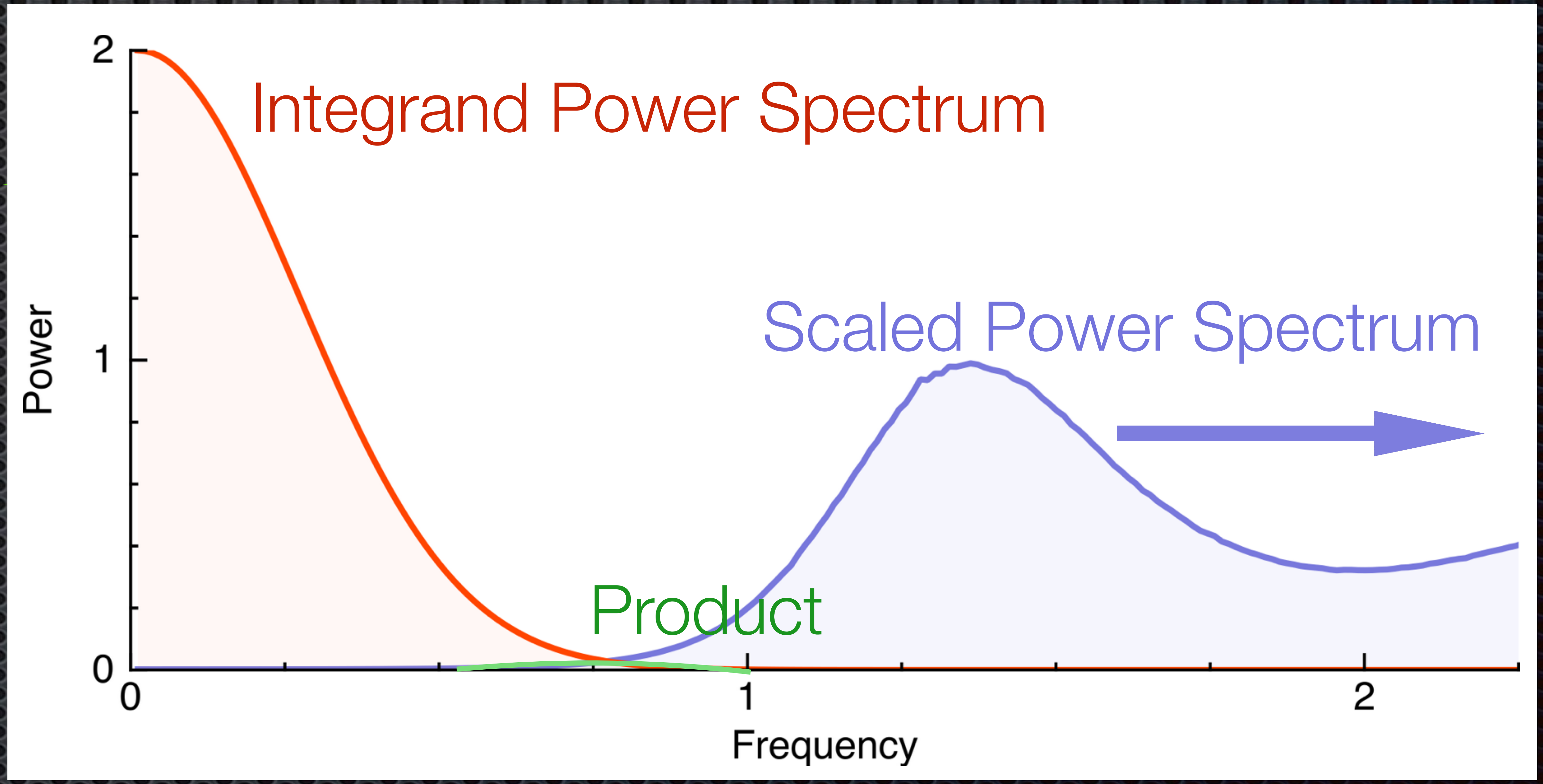


For a given number of samples

Dependence on Number of Samples



For a given number of samples



Increasing the number of samples

$$Var(\boldsymbol{I}_N) = \frac{\mu(\mathcal{T}^d)\mu(S^{d-1})}{N} \int_0^\infty \rho^{d-1} \breve{\mathcal{P}}_{\mathbf{S}}(\rho) \breve{\mathcal{P}}_{\mathbf{F}}(\rho) \mathrm{d}\rho$$

Euclidean

$$Var(\boldsymbol{I}_N) = \frac{\mu(\mathcal{T}^d)\mu(S^{d-1})}{N} \int_0^\infty \rho^{d-1} \breve{\mathcal{P}}_{\mathbf{S}}(\rho) \breve{\mathcal{P}}_{\mathbf{F}}(\rho) \mathrm{d}\rho$$

Euclidean

$$\breve{\mathcal{P}}_{\mathbf{F}}(\rho)$$

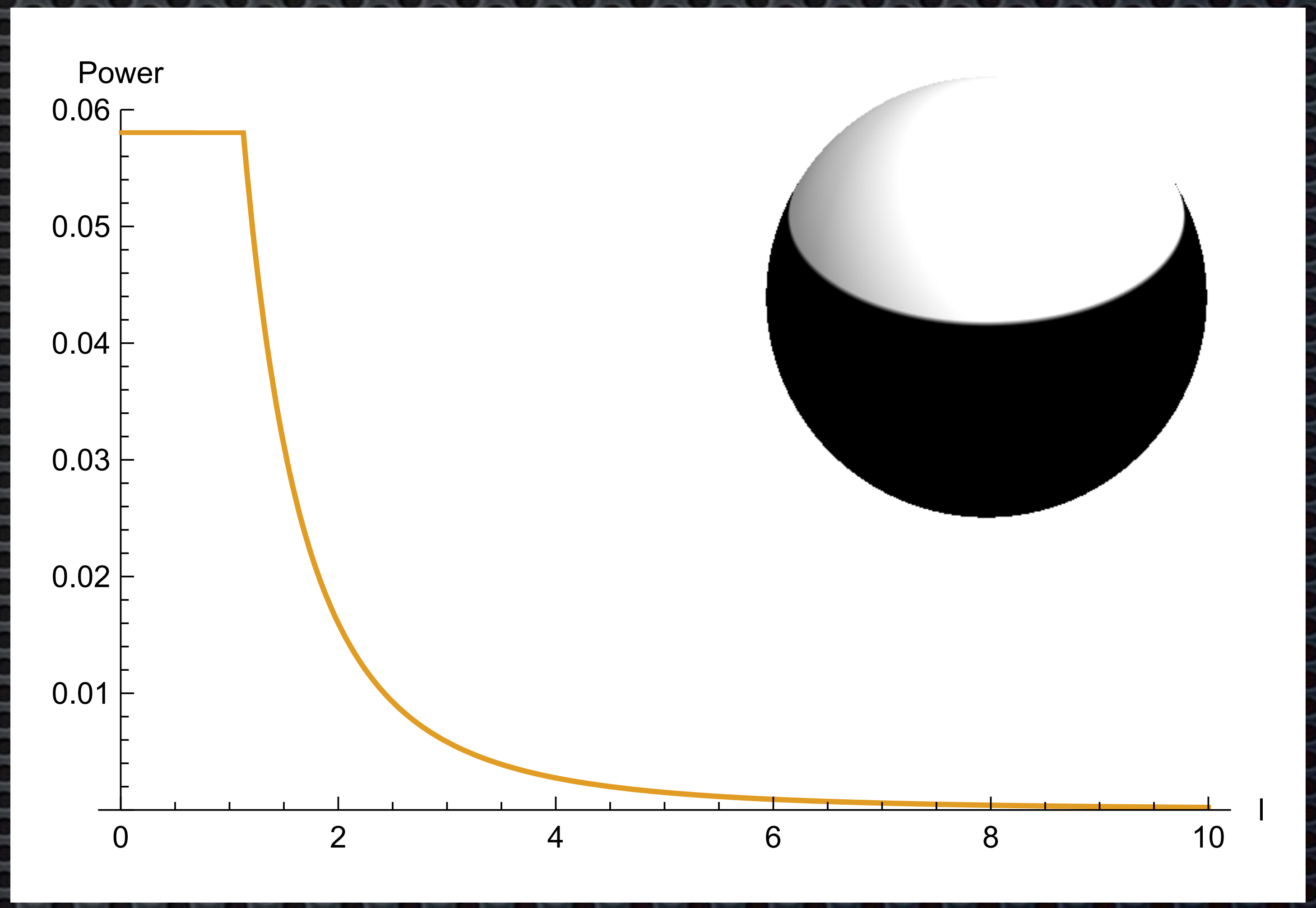
$$\underbrace{Var(\boldsymbol{I}_N)}_{\text{Euclidean}} = \frac{\mu(\mathcal{T}^d)\mu(S^{d-1})}{N} \int_0^\infty \rho^{d-1} \check{\mathcal{P}}_{\mathbf{S}}(\rho) \check{\mathcal{P}}_{\mathbf{F}}(\rho) \mathrm{d}\rho$$

$$\check{\mathcal{P}}_{\mathbf{S}}(\rho) \qquad \check{\mathcal{P}}_{\mathbf{F}}(\rho)$$

Integrand Power Spectrum

$$\breve{\mathcal{P}}_{\mathbf{F}}(\rho) = \begin{cases} c_F & \rho < \rho_0 \\ c'_F \rho^{-d-1} & \text{otherwise} \end{cases}$$

where, c_F and c'_F are constants

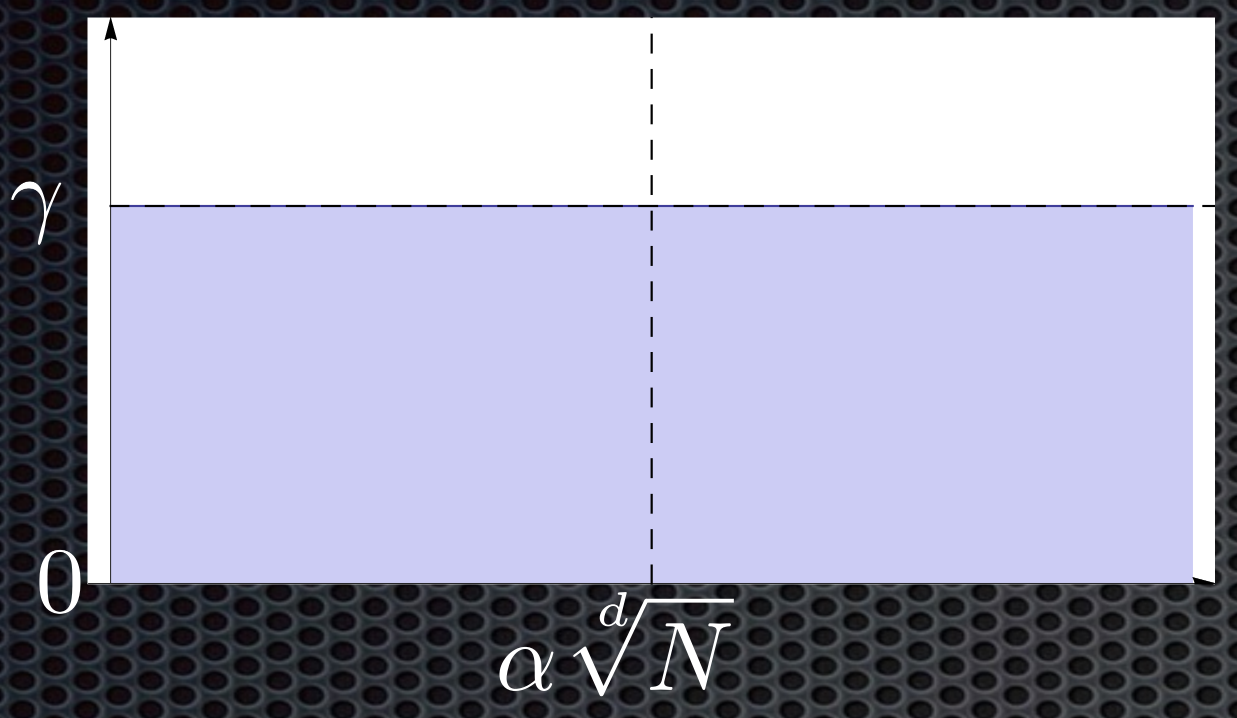


[Brandolini et al 2001, Mean square decay of Fourier transforms in euclidean and non euclidean spaces]

Theoretical Convergence Analysis

$b = 0$

$\breve{\mathcal{P}}_{\mathbf{S}}(\rho)$



$Var(\boldsymbol{I}_N)$

$$O\left(\frac{1}{N}\right)$$

b = degree of the polynomial

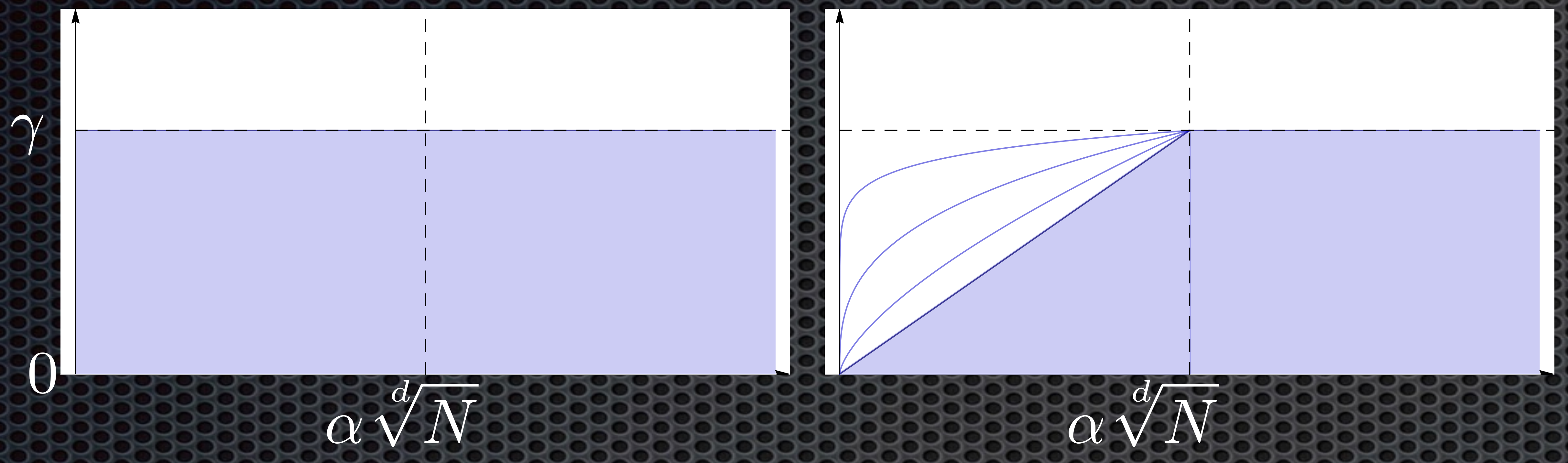
d = dimensions

N = number of samples

Theoretical Convergence Analysis

$b = 0$ $\quad\quad$ $0 < b \leq 1$

$\breve{\mathcal{P}}_S(\rho)$



$Var(I_N)$

$$O\left(\frac{1}{N}\right) \qquad O\left(\frac{1}{N\sqrt[d]{N}^{b}}\right)$$

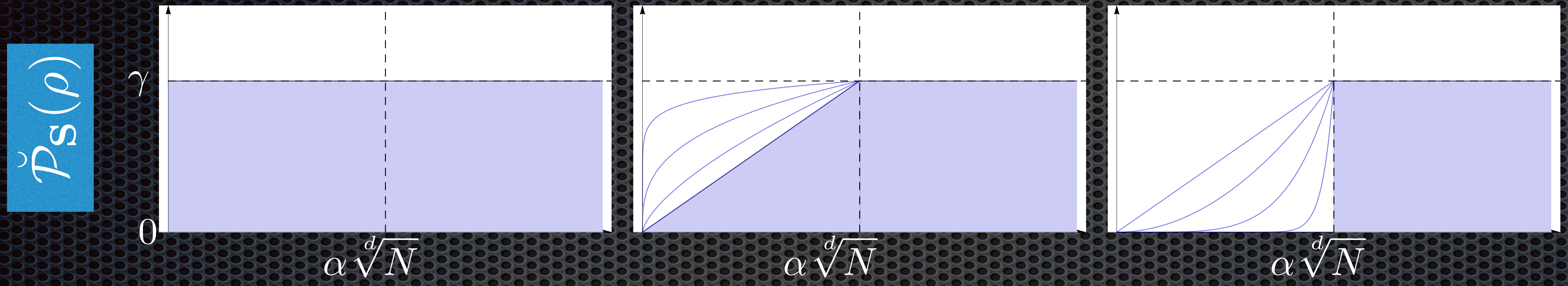
b = degree of the polynomial

d = dimensions

N = number of samples

Theoretical Convergence Analysis

$b = 0$ $\qquad$ $0 < b \leq 1$ $\qquad$ $b \geq 1$



$Var(\boldsymbol{I}_N)$

$$O\left(\frac{1}{N}\right) \qquad O\left(\frac{1}{N\sqrt[d]{N}^{b}}\right) \qquad O\left(\frac{1}{N\sqrt[d]{N}}\right)$$

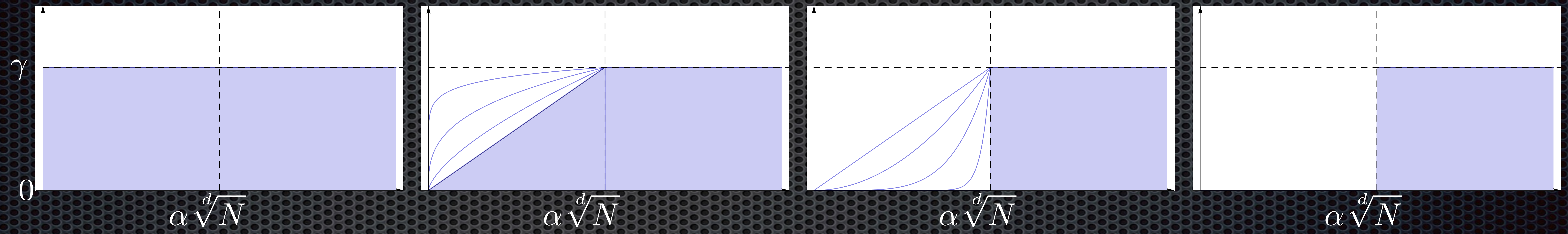
b = degree of the polynomial

d = dimensions

N = number of samples

Theoretical Convergence Analysis

	$b = 0$	$0 < b \leq 1$	$b \geq 1$	$b \to \infty$
$\breve{\mathcal{P}}_S(\rho)$				

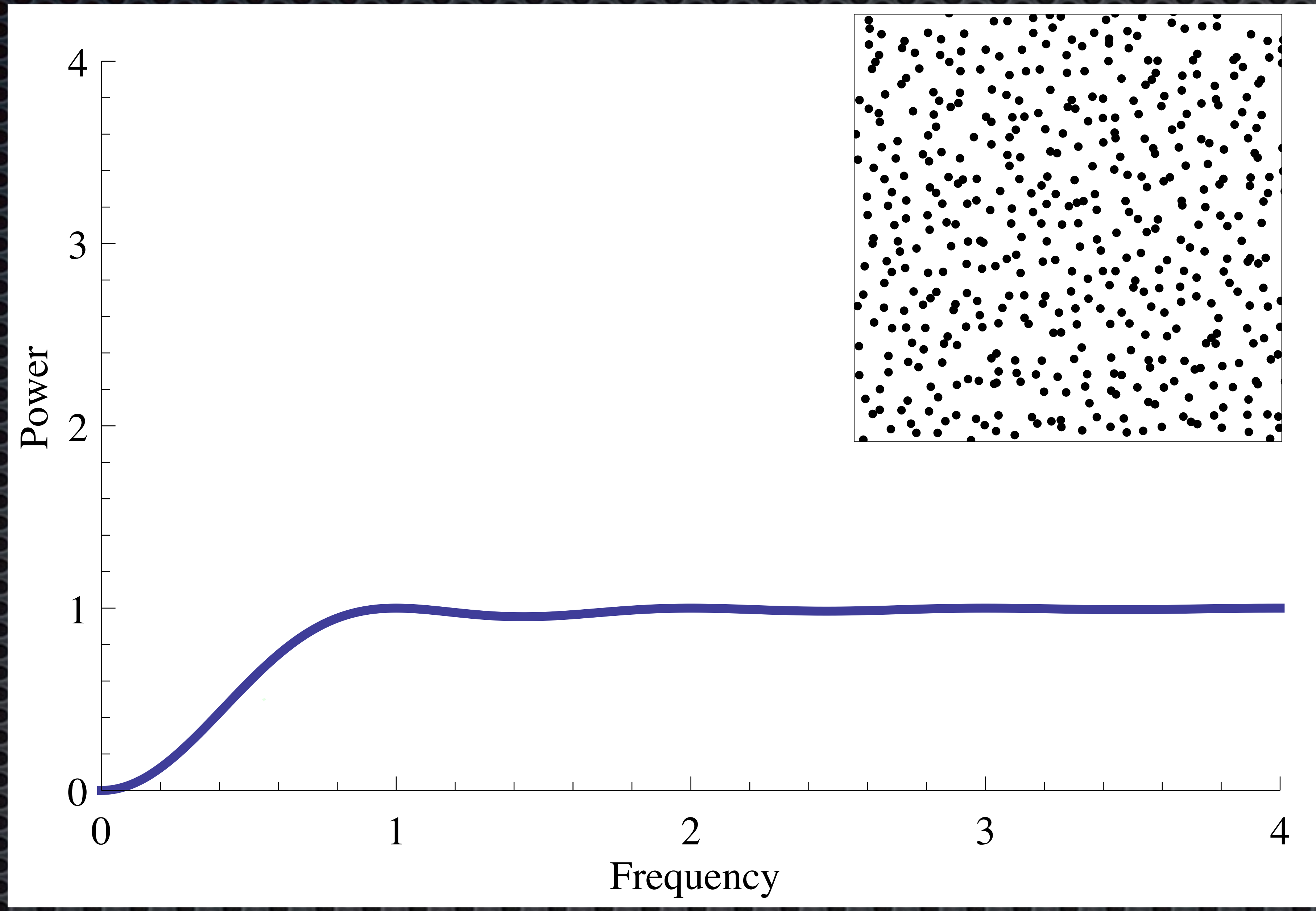


	$b = 0$	$0 < b \leq 1$	$b \geq 1$	$b \to \infty$
$Var(I_N)$	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N\sqrt[d]{N}^{b}}\right)$	$O\left(\frac{1}{N\sqrt[d]{N}}\right)$	$O\left(\frac{1}{N\sqrt[d]{N}}\right)$

b = degree of the polynomial d = dimensions N = number of samples

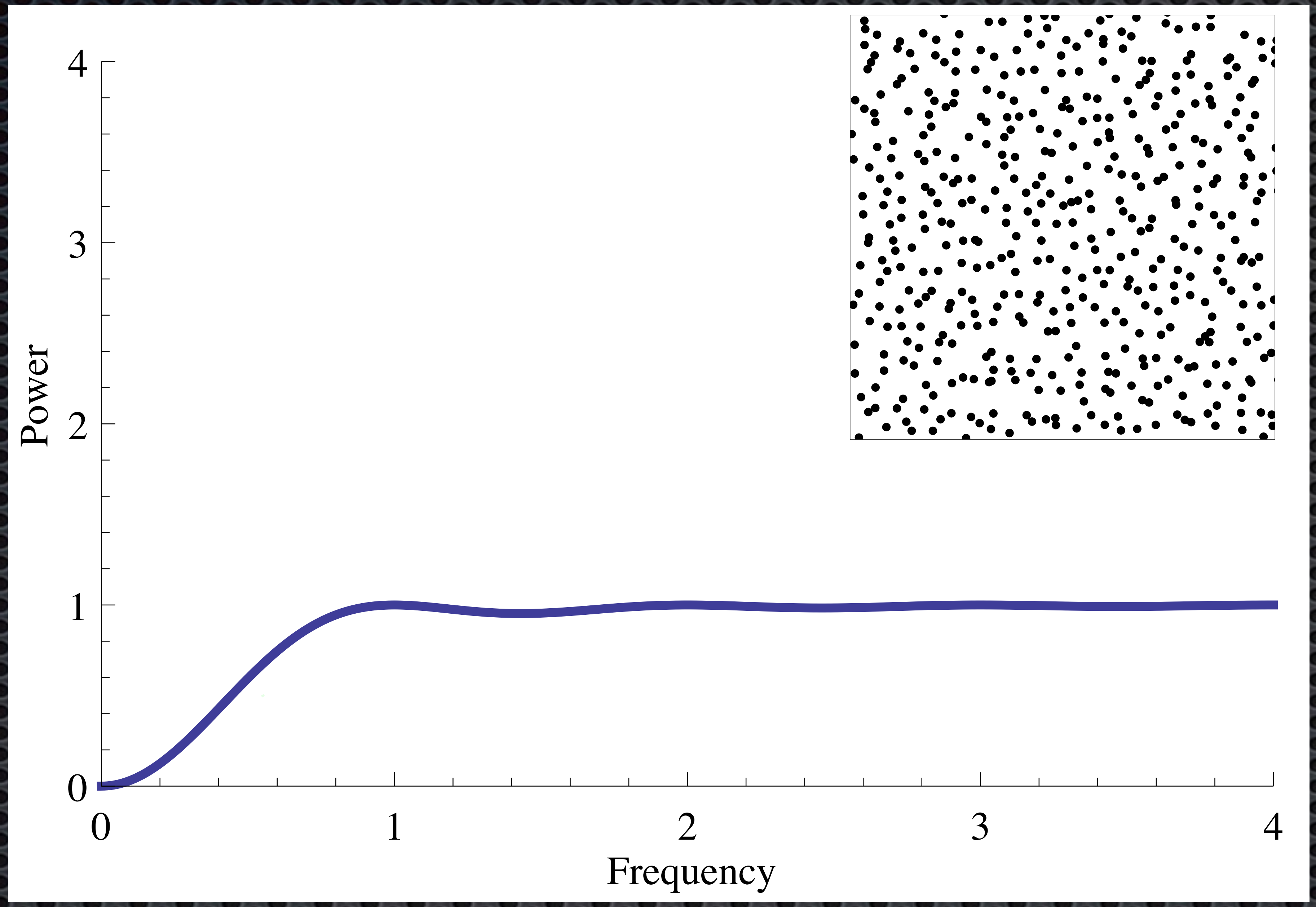
Power Spectrum Bounds

Jittered

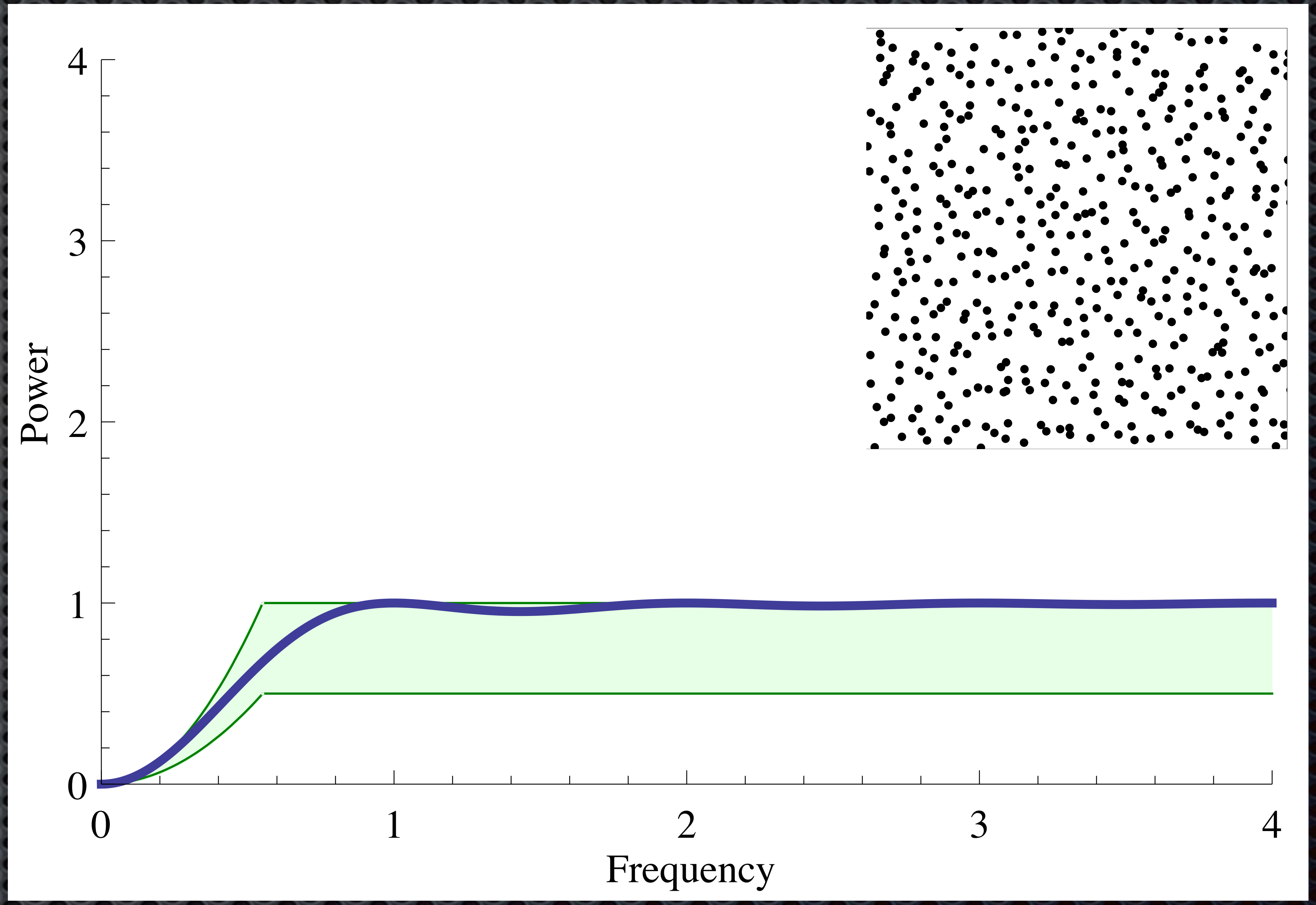


Power Spectrum Bounds

Jittered



With Bounds

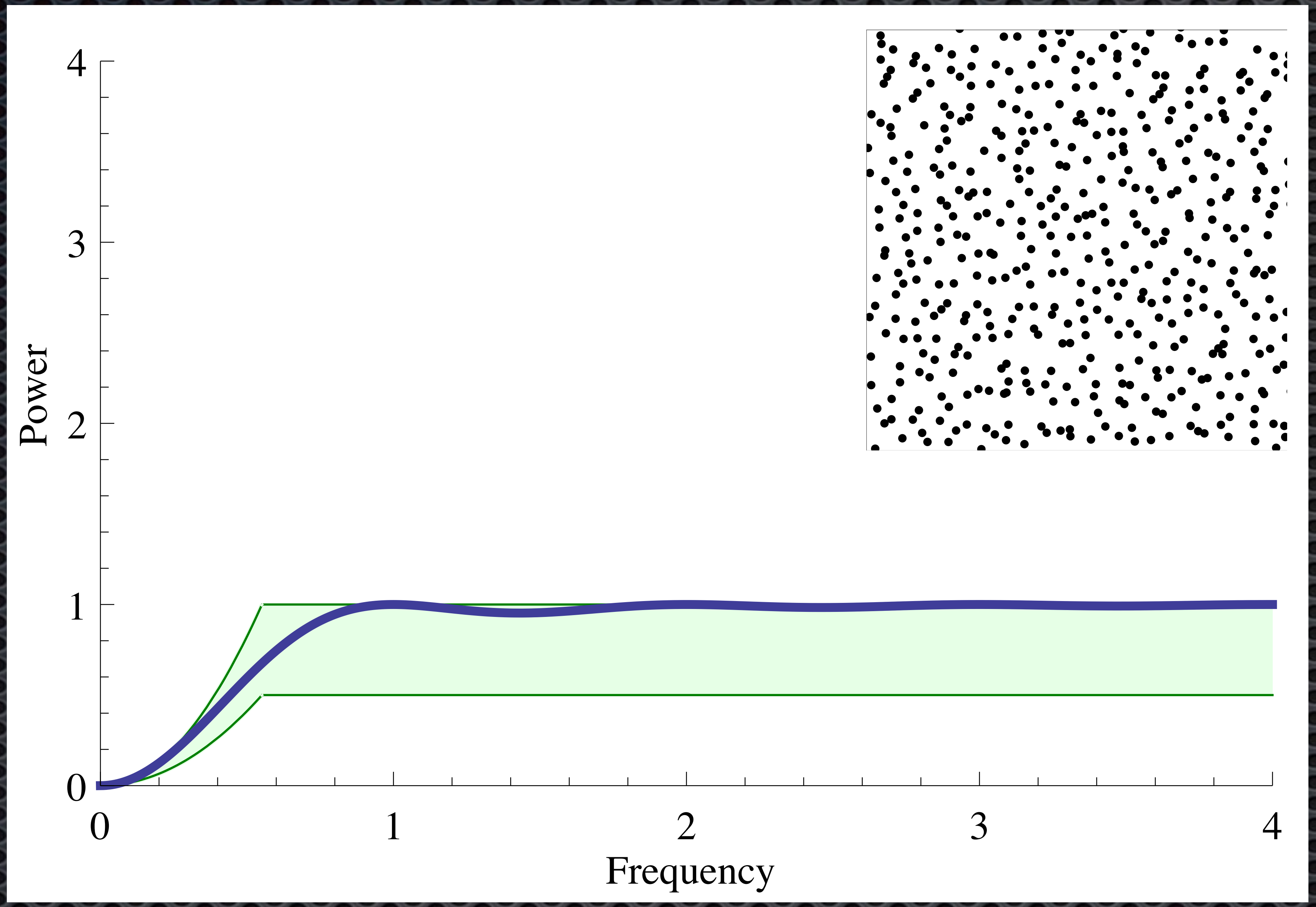


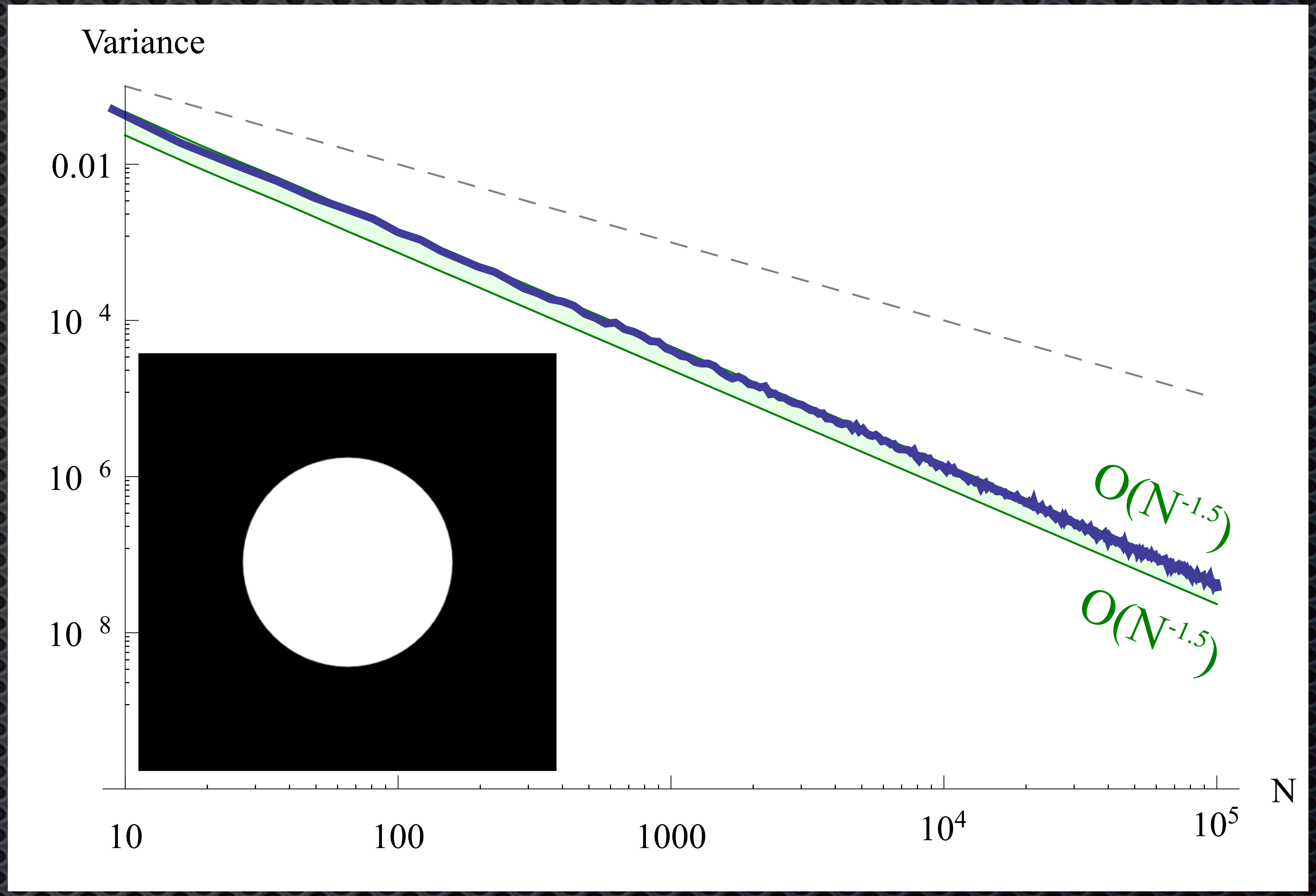
Convergence Rate Analysis

Power Spectrum

Convergence rate

Jittered



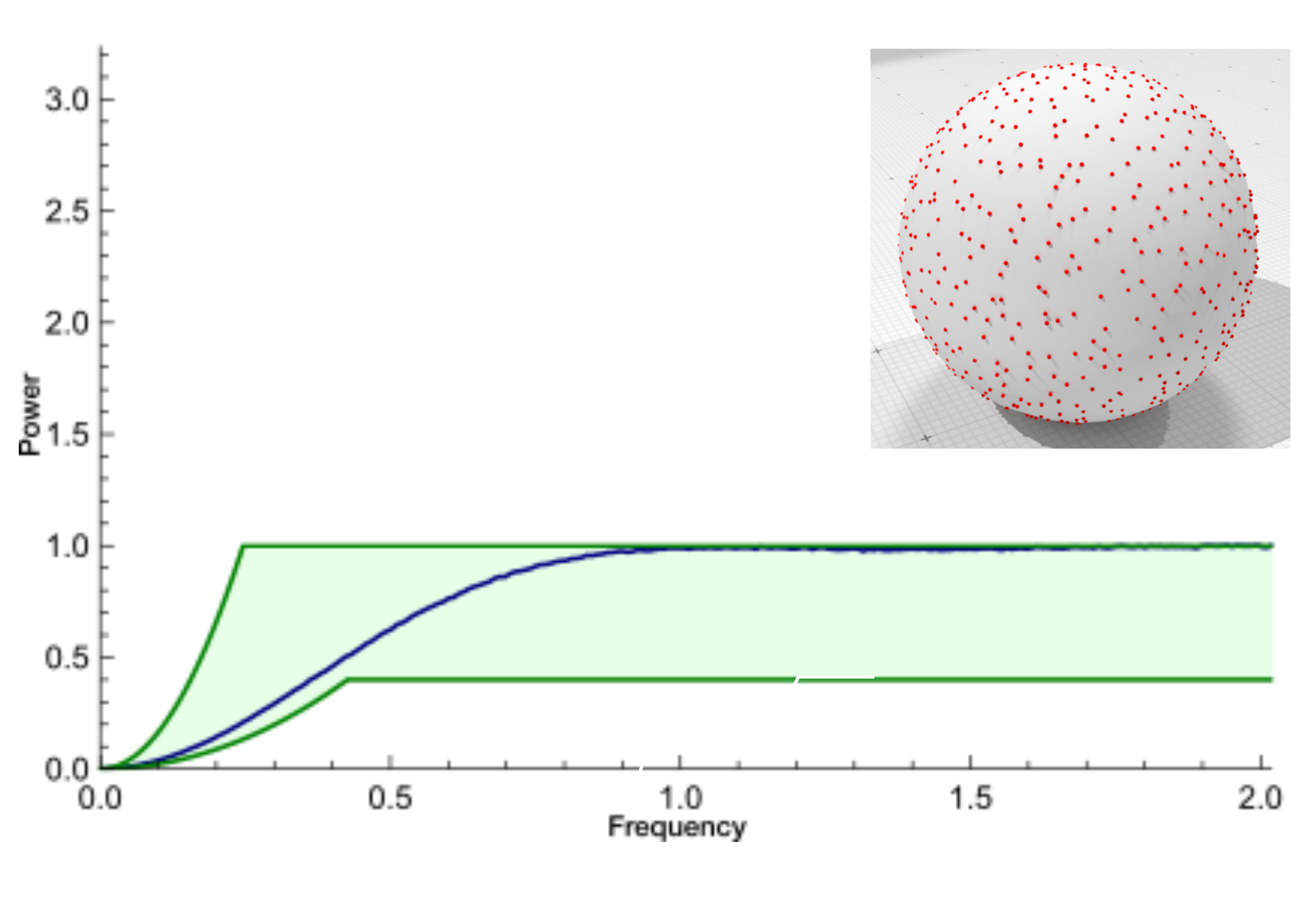


Variance convergence rate: $O\left(\frac{1}{N\sqrt{N}}\right)$

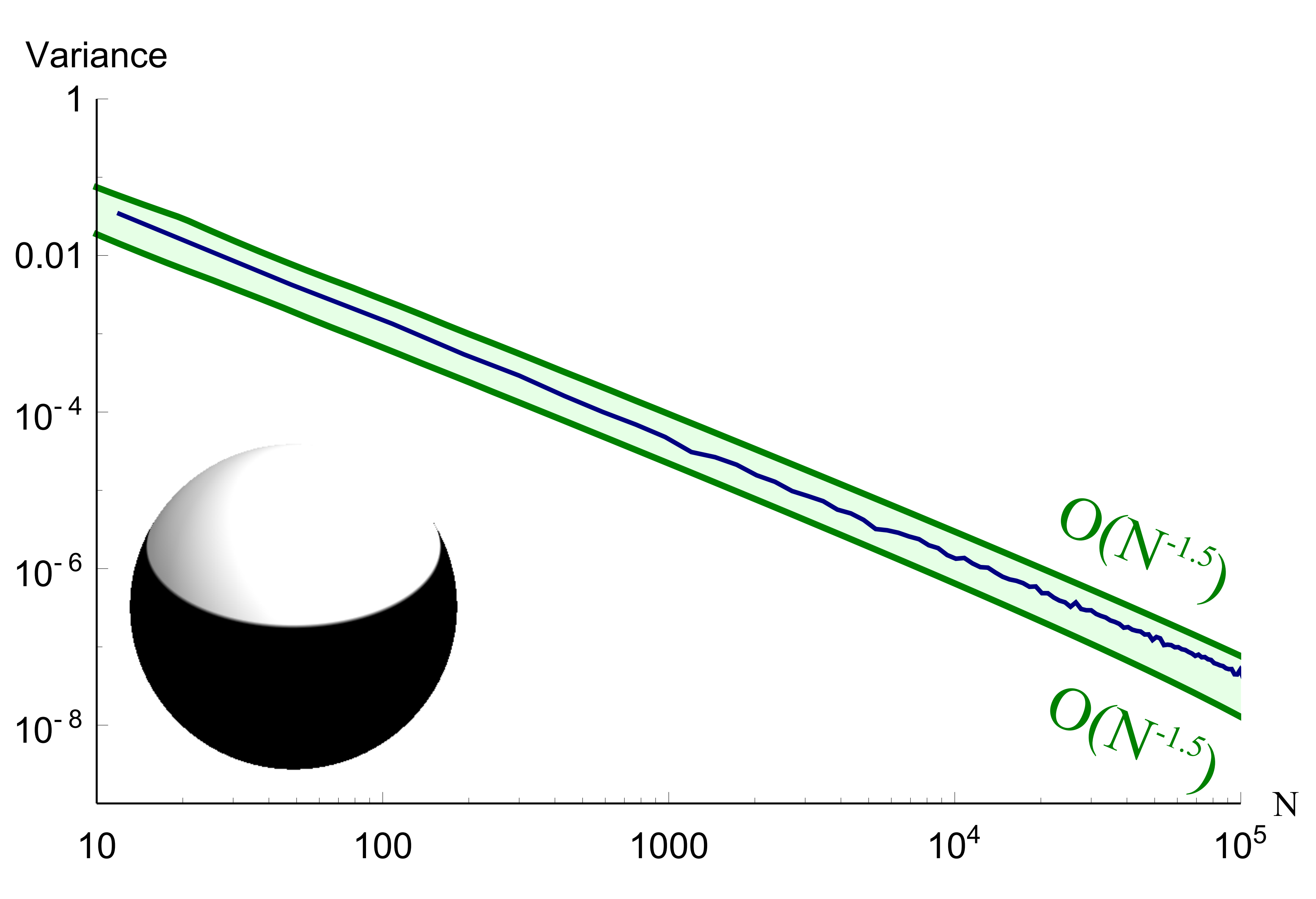
Convergence Rate Analysis

Jittered

Power Spectrum



Convergence rate



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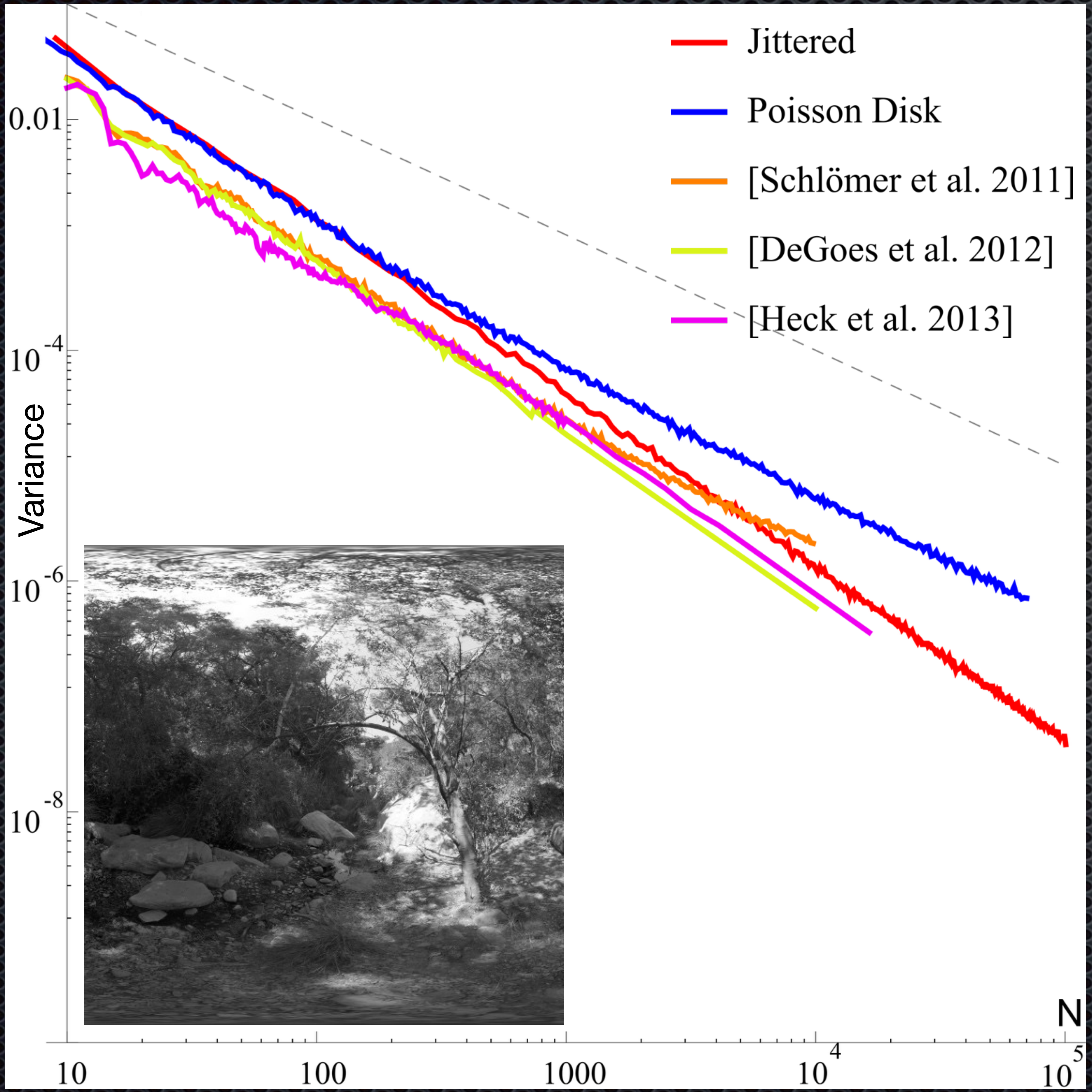
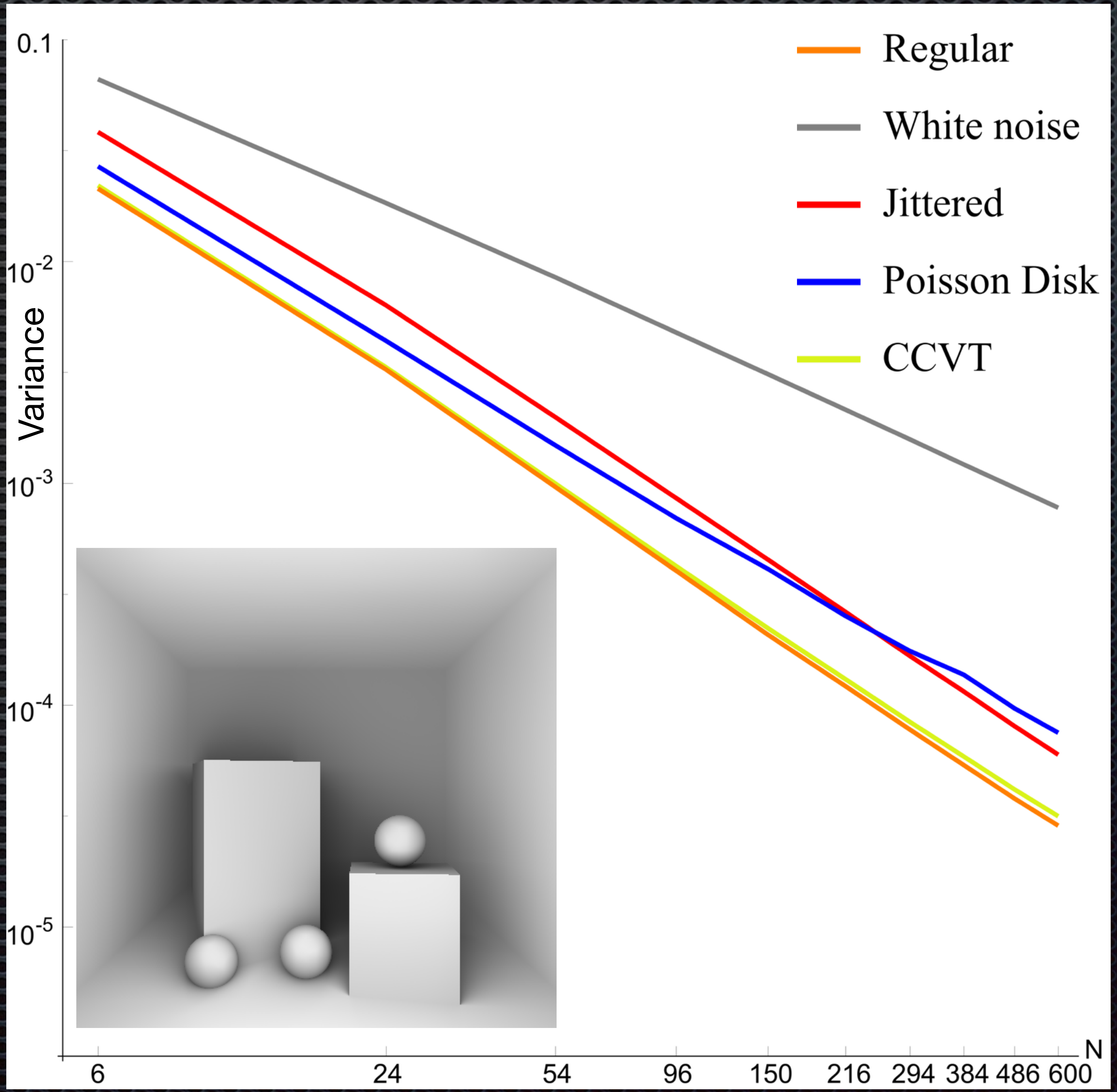
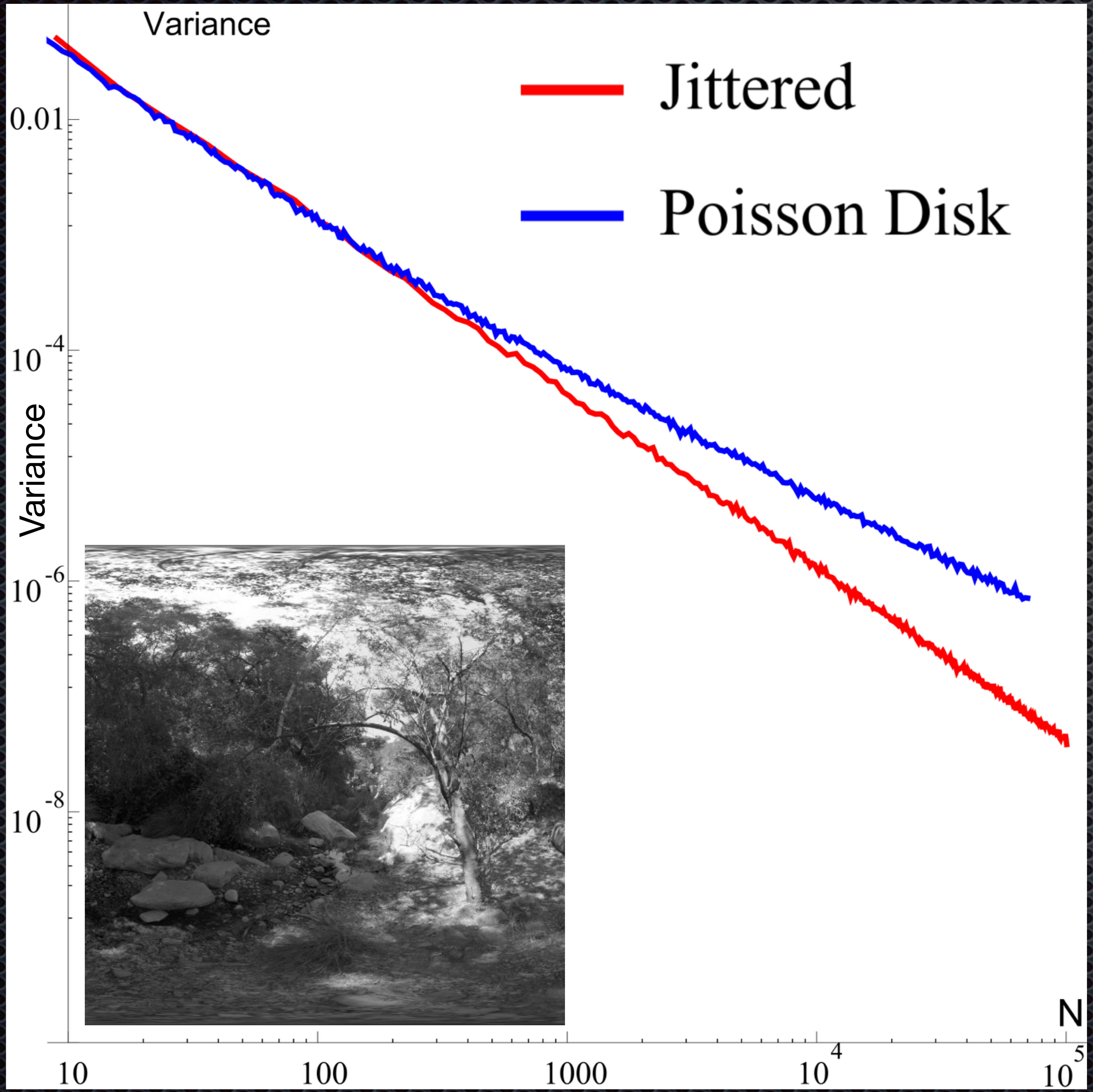
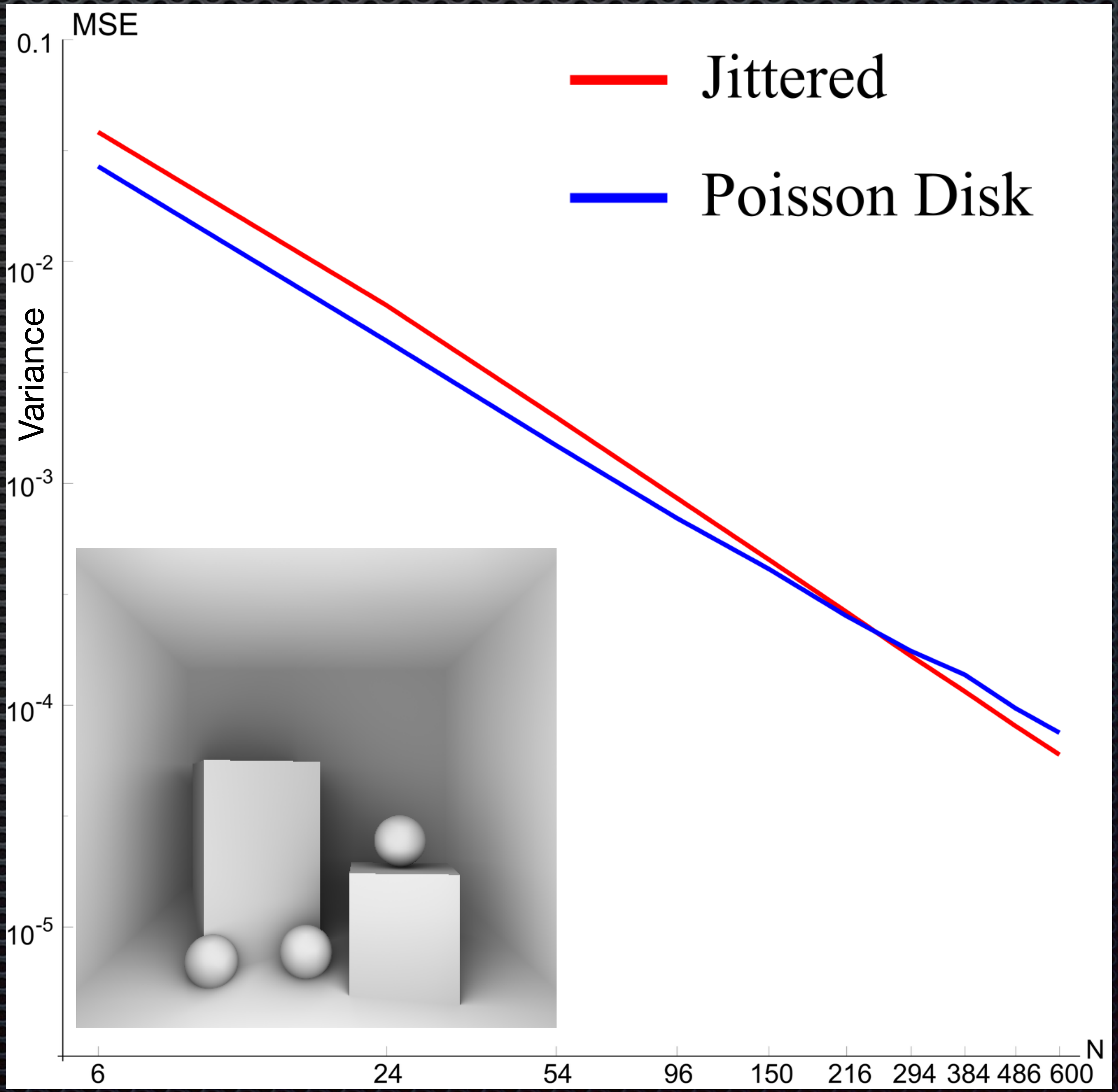
Convergence Rate Analysis





Jittered vs Poisson Disk Sampling

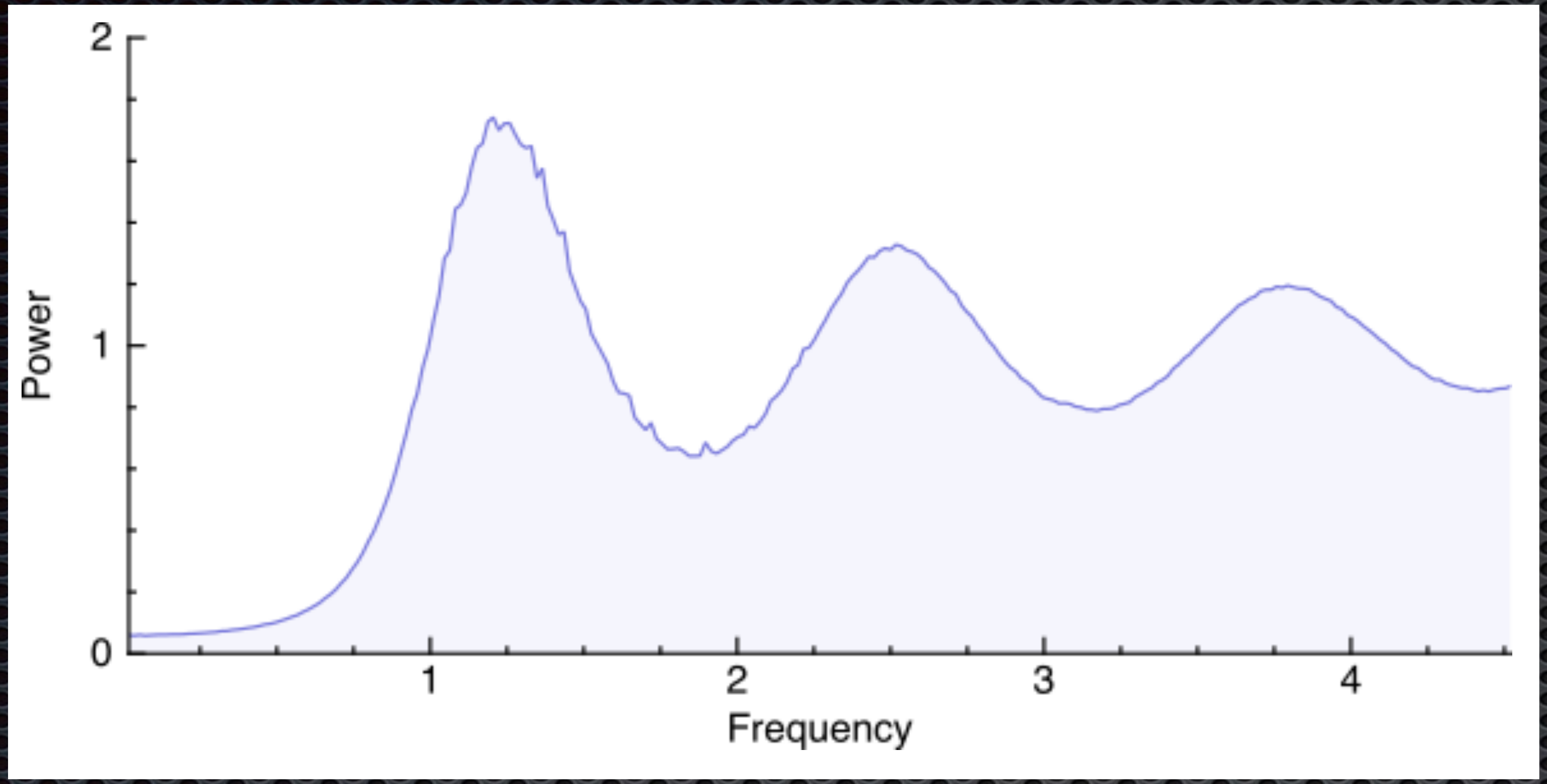


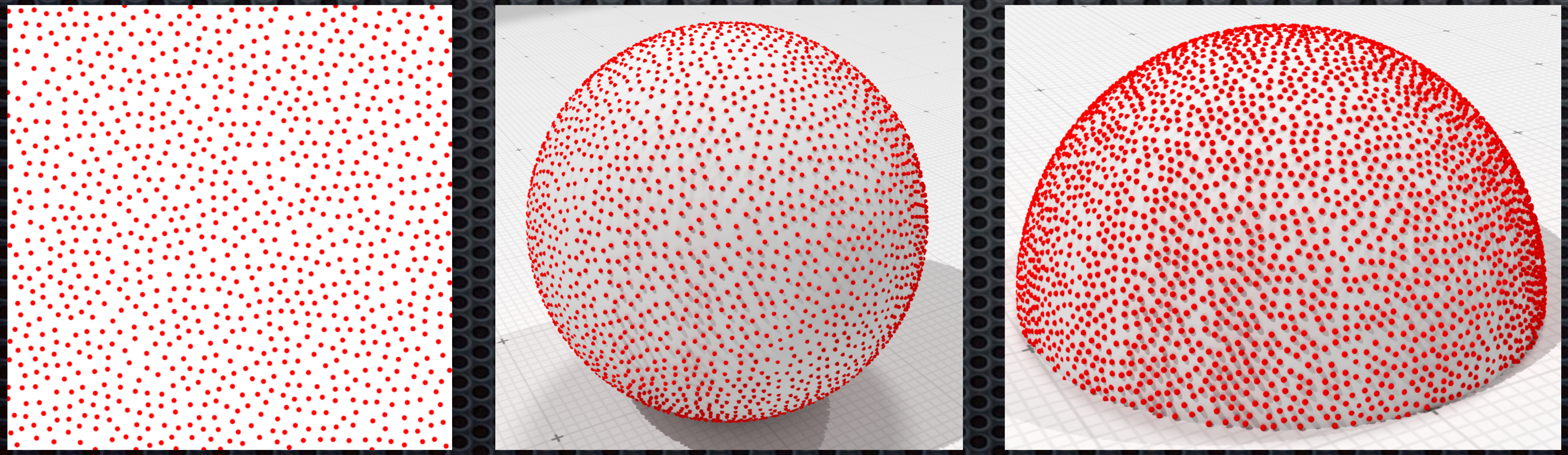


Why is jittered sampling
better than Poisson Disk sampling ?

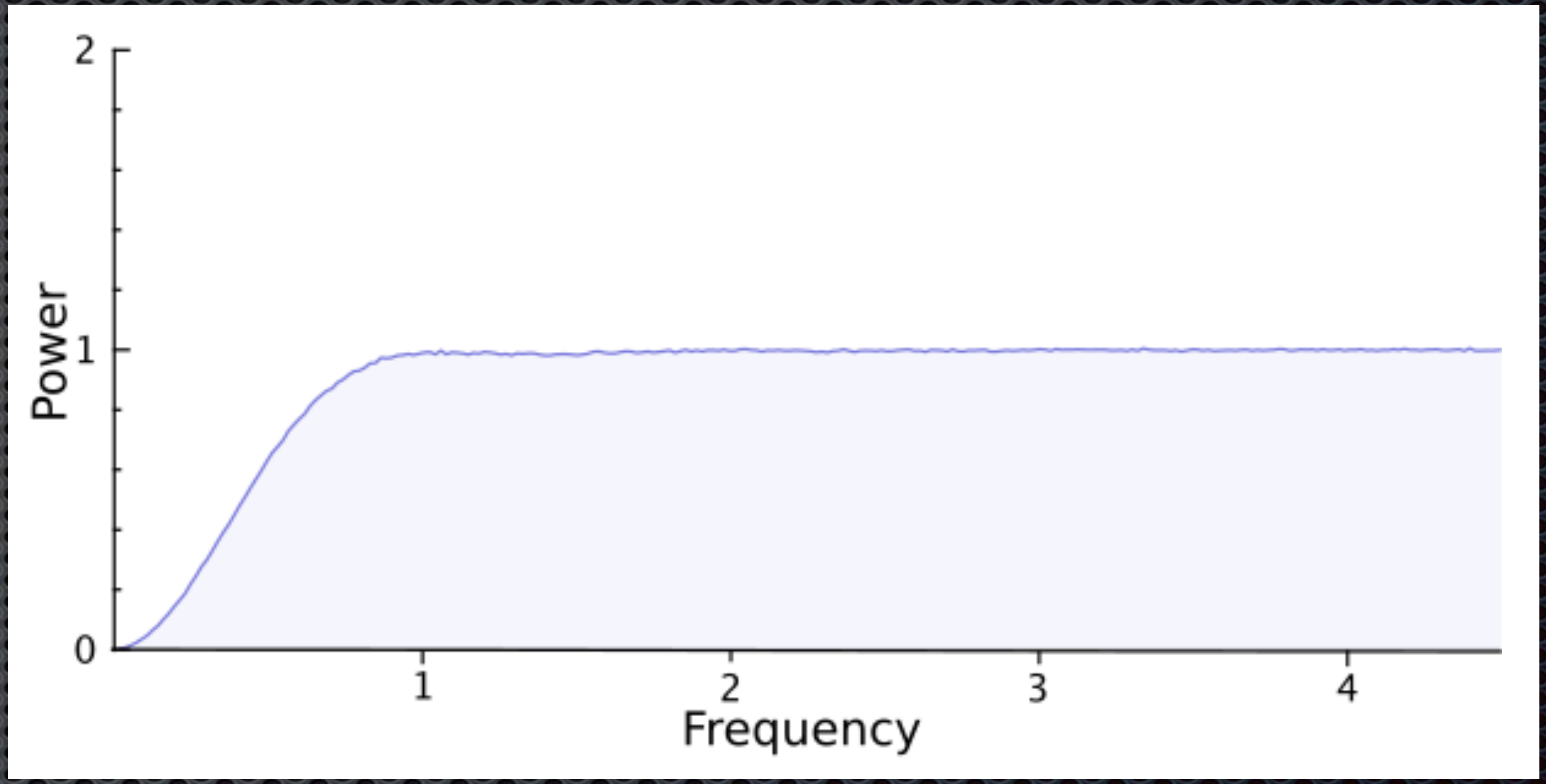
Power Spectra

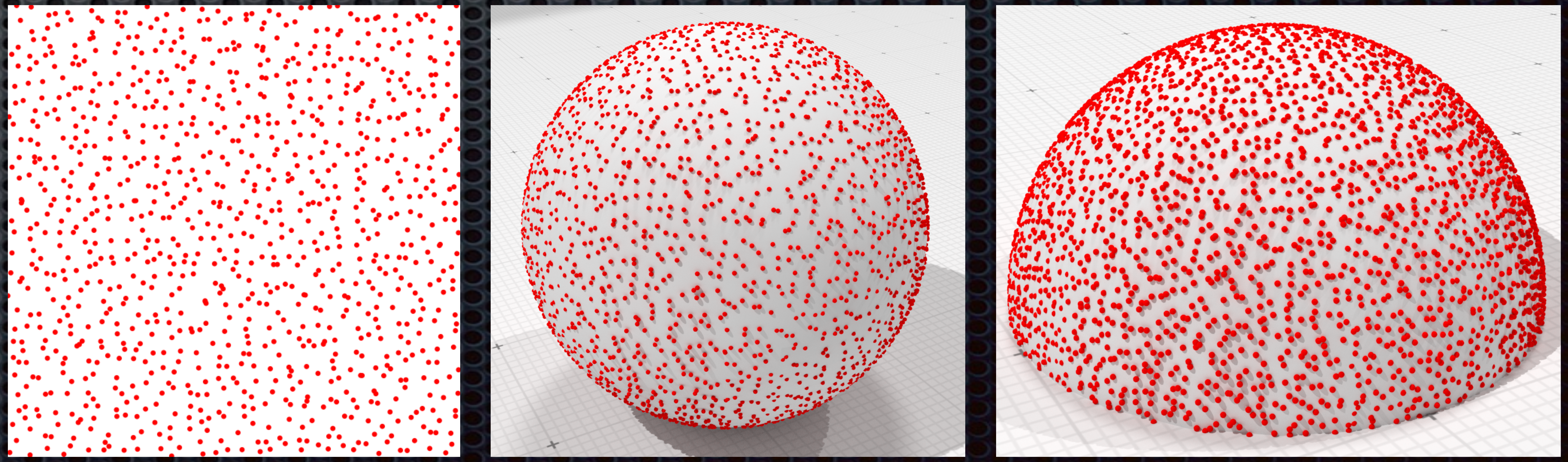
Poisson Disk





Jittered

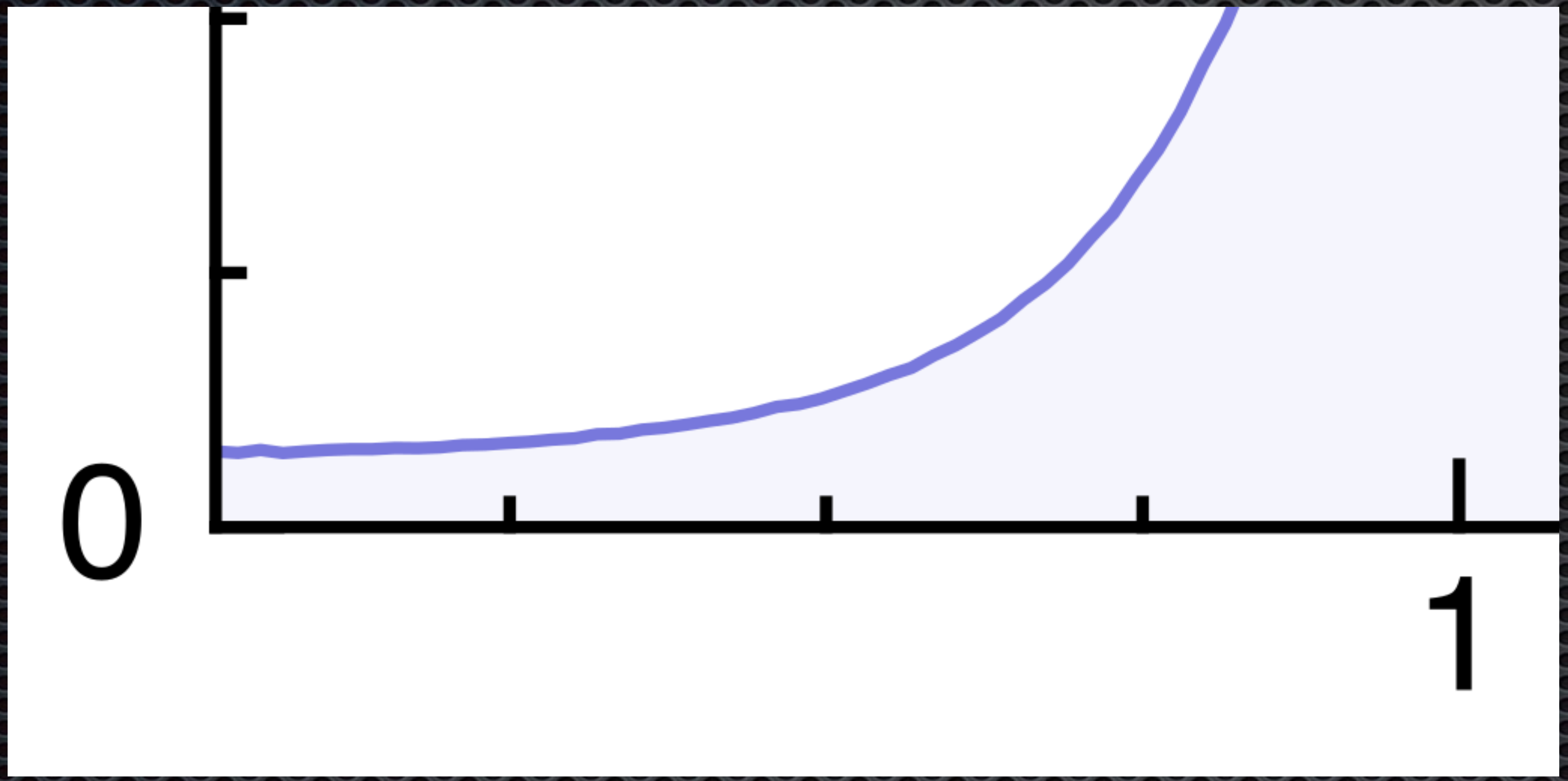




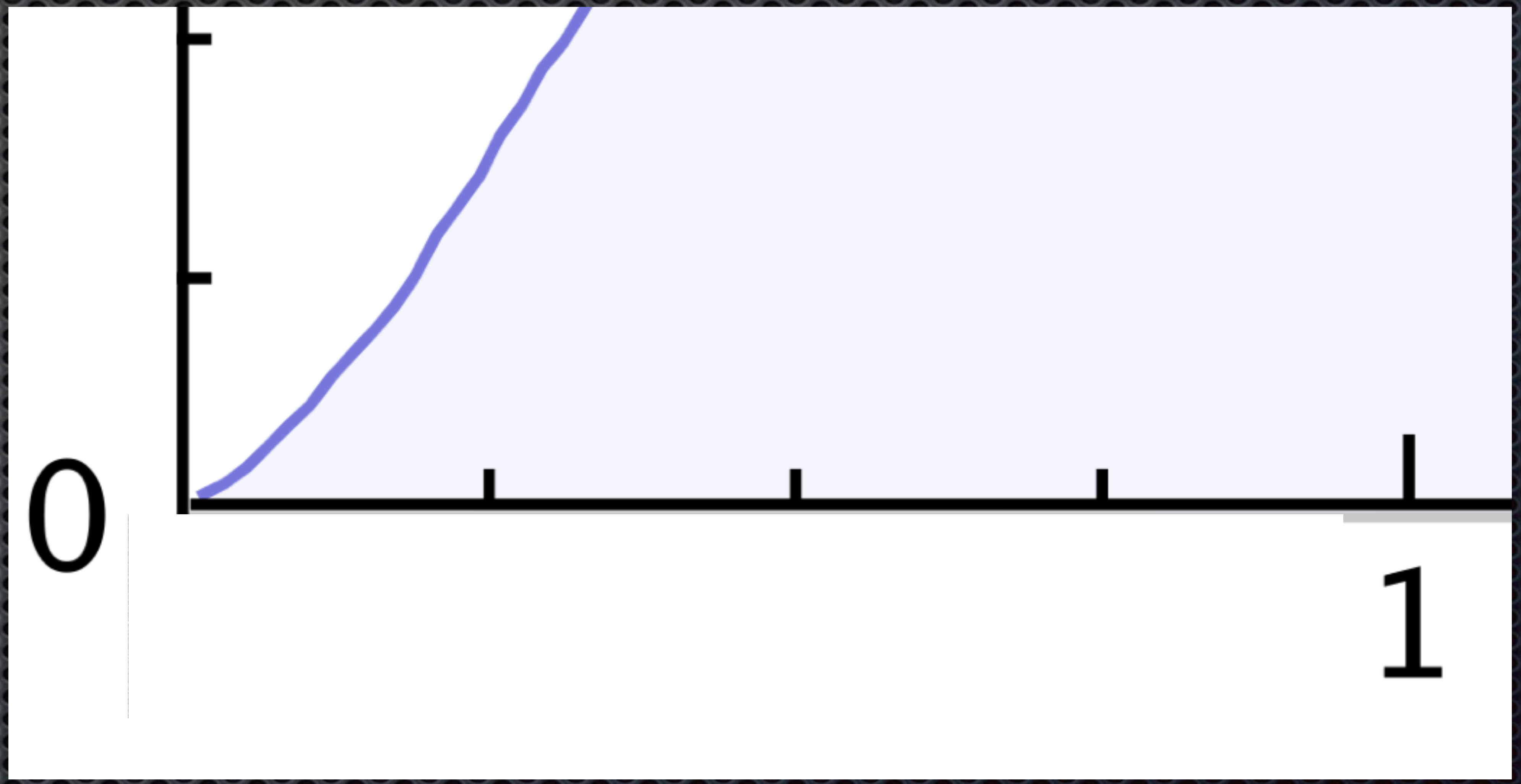
Power Spectra: Low Frequency Region

Poisson Disk $O\left(\frac{1}{N}\right)$

Jittered $O\left(\frac{1}{N\sqrt{N}}\right)$

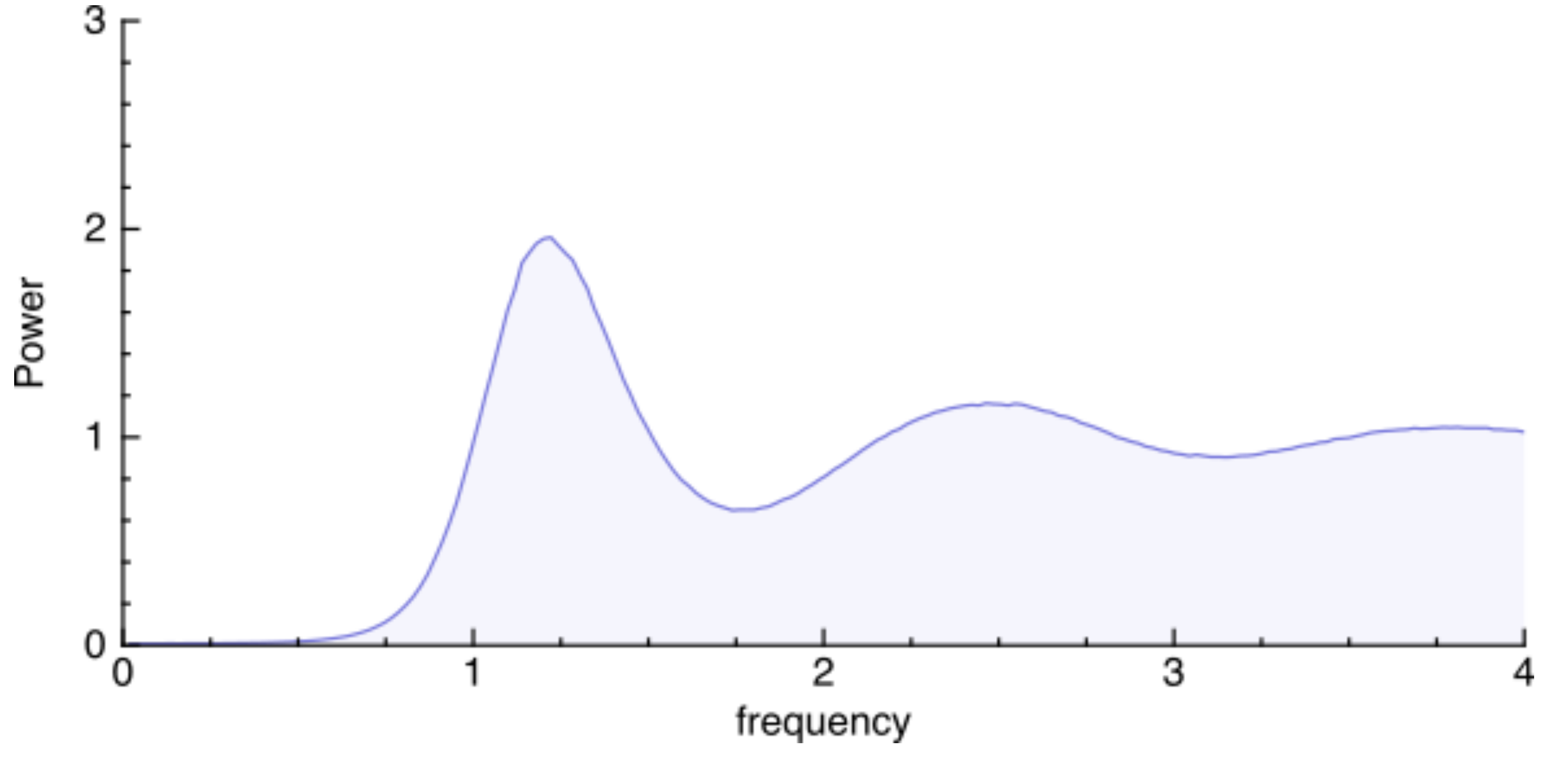


Constant Offset



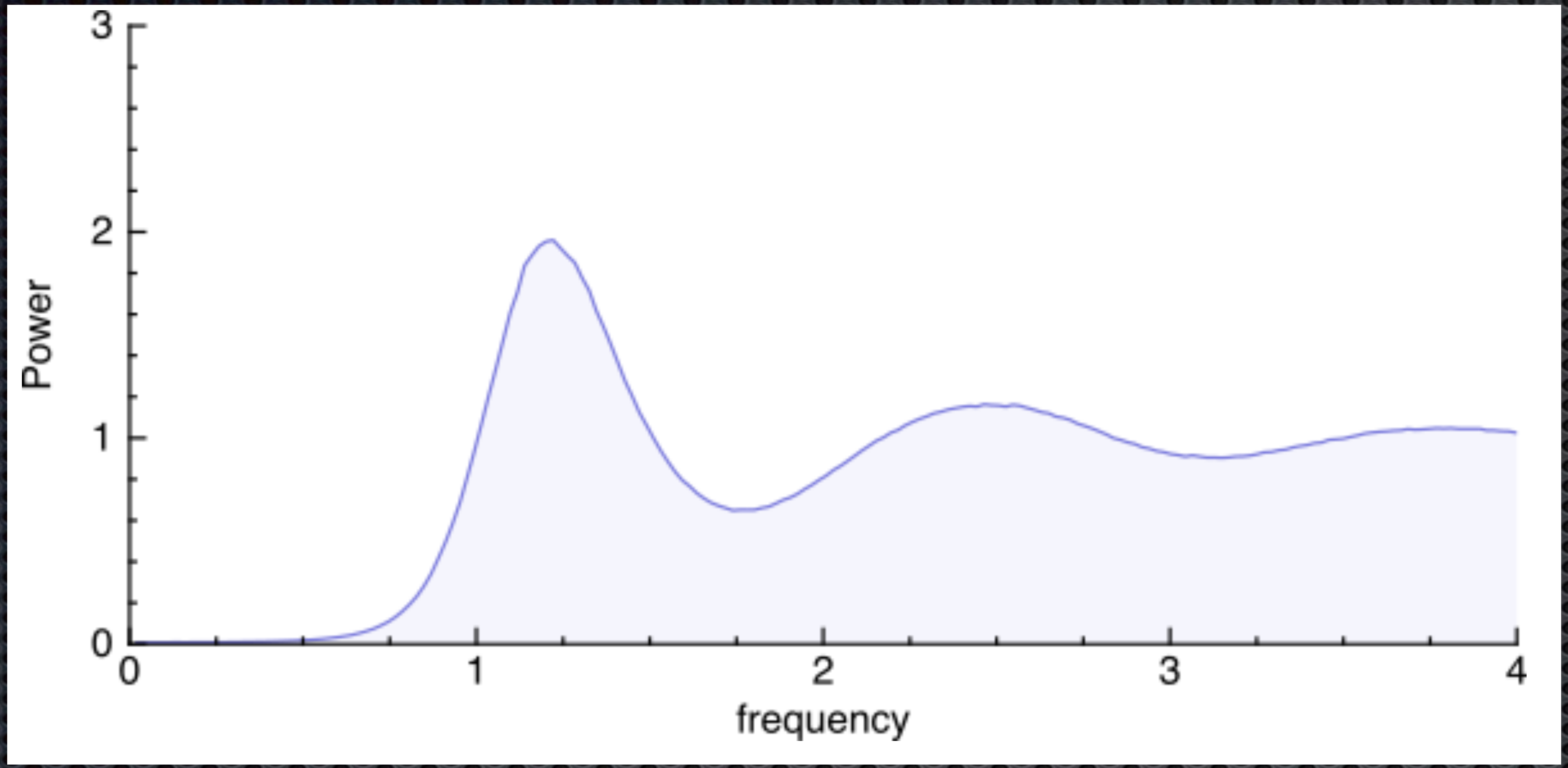
Approaching Zero

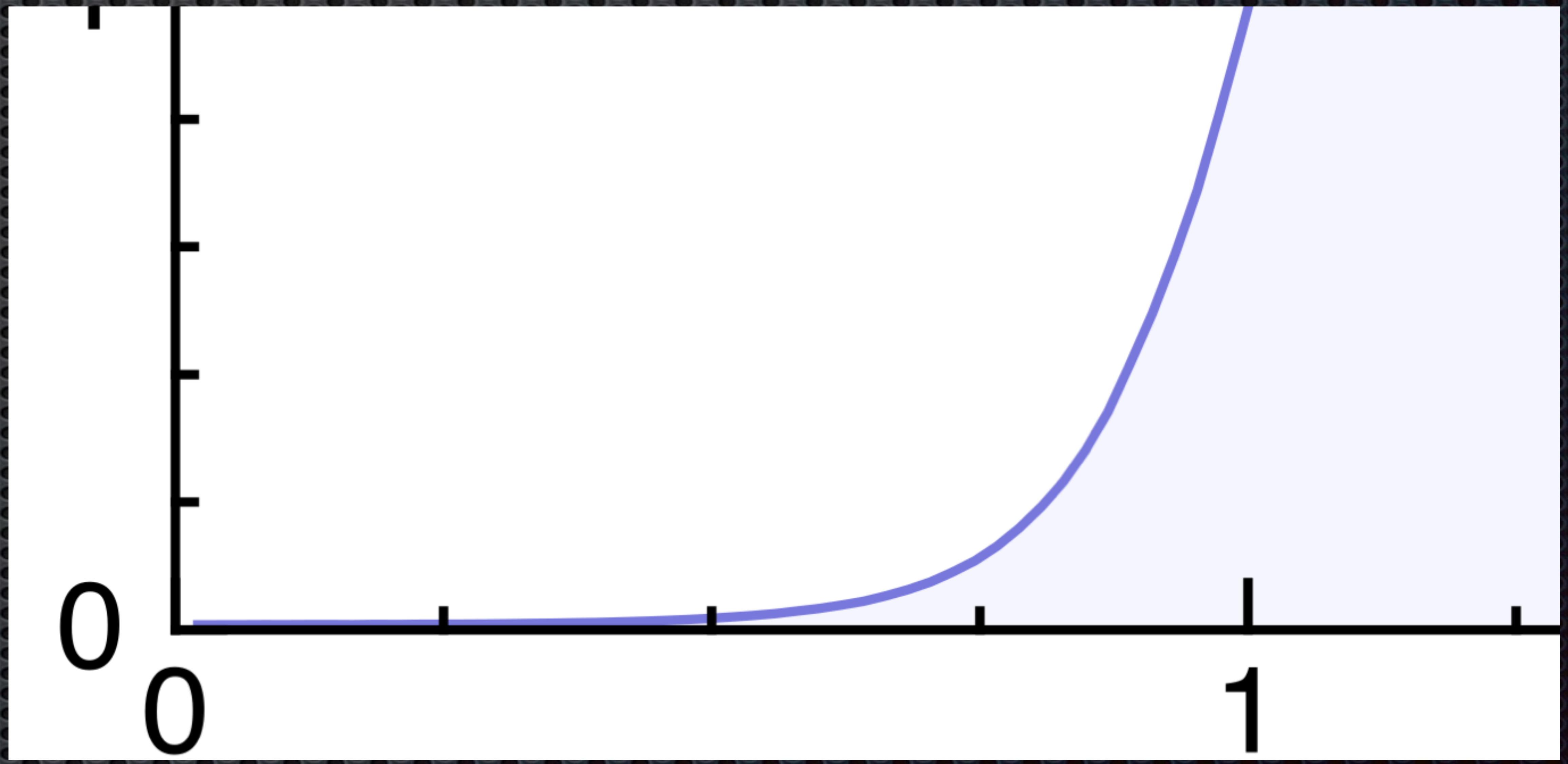
CCVT *[Balzer et al. 2009]*



Variance Convergence Rate: $O\left(\frac{1}{N\sqrt{N}}\right)$

CCVT *[Balzer et al. 2009]*





Variance Convergence Rate: $O\left(\frac{1}{N\sqrt{N}}\right)$

Our mathematical model can be used
to tailor new sampling patterns.

Novel Contributions

- Frequency analysis of spherical and hemispherical samples using spherical harmonics

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- Unified closed form variance expression that can be used to design new sampling patterns

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- Unified closed form variance expression that can be used to design new sampling patterns
- Analysis tool to theoretically compute and bound variance convergence rates of any stochastic sampler

Future Work

- Extend our mathematical framework to adaptive sampling strategies

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- Explore how we can extend our mathematical model to deterministic sampling patterns

Future Work

- Extend our mathematical framework to adaptive sampling strategies
- Explore how we can extend our mathematical model to deterministic sampling patterns
- Use our framework to construct new sampling patterns with the best convergence speed and with lowest variance even for small number of samples

Our tools will be made public very soon.

http://liris.cnrs.fr/variance

Acknowledgements

- ANR excellence chair (ANR-10-CEXC-002-01)
- DigitalSnow/DigitalFoam programs(ANR-11-BS02-009 and PALSE/2013/21)

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- Kartic Subr
- Mathieu Desbrun and Katherine Breeden
- Jonathan Dupuy and Nicolas Bonneel
- Jean-Claude Iehl, Vincent Nivoliers and Brian Wyvill
- Anonymous reviewers

Thank you for your attention.

$$Var(\boldsymbol{I}_N) \propto \frac{1}{N} \int_0^\infty \rho^{d-1} \breve{\mathcal{P}}_{\mathbf{S}}(\rho) \breve{\mathcal{P}}_{\mathbf{F}}(\rho) \mathrm{d}\rho$$

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Power spectra behavior
at low frequencies
matters !

Thank you for your attention.

$$Var(\boldsymbol{I}_N) \propto \frac{1}{N}\int_0^{\infty} \rho^{d-1} \breve{\mathcal{P}}_{\mathbf{S}}(\rho)\breve{\mathcal{P}}_{\mathbf{F}}(\rho)\,\mathrm{d}\rho$$

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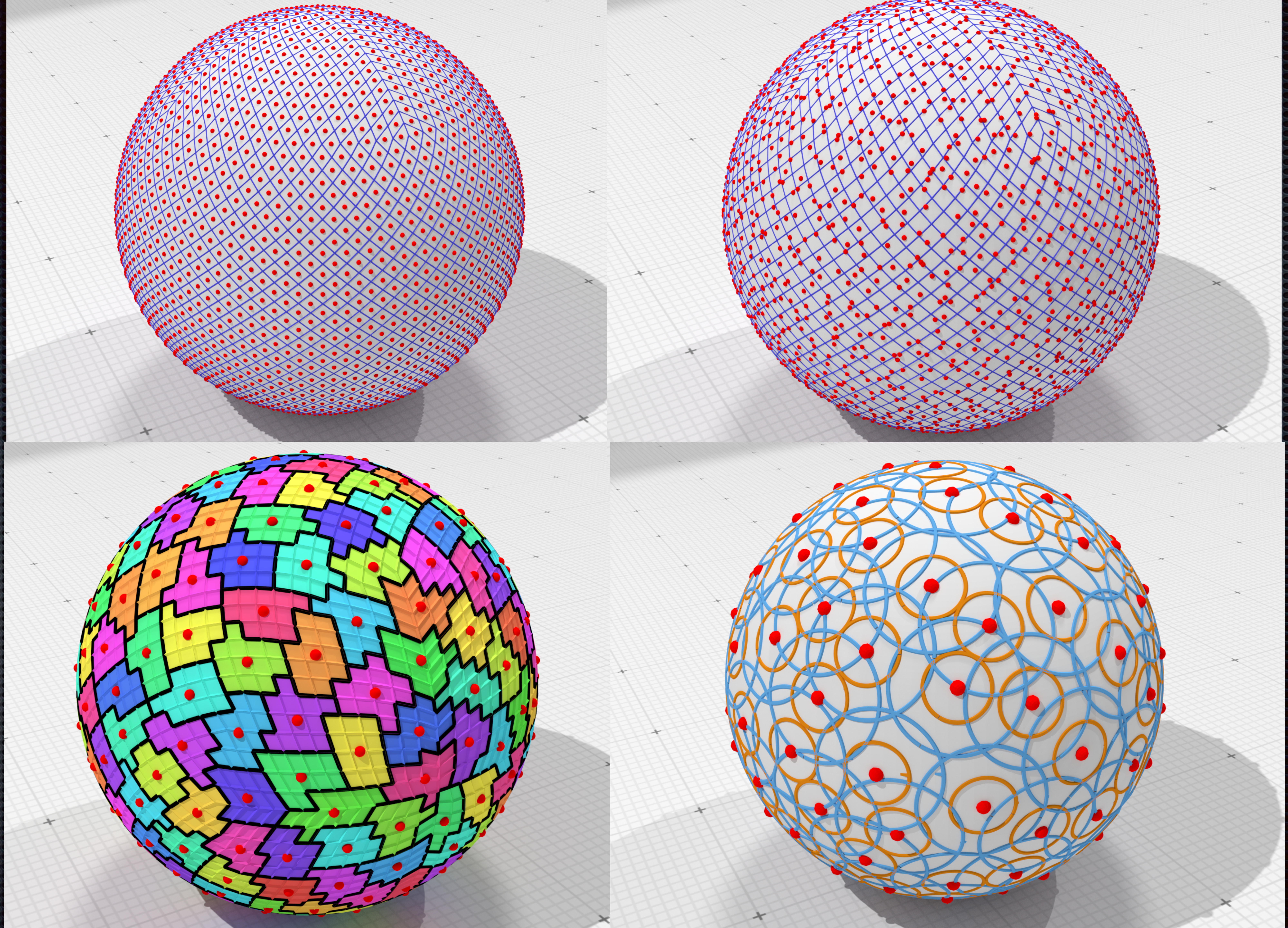
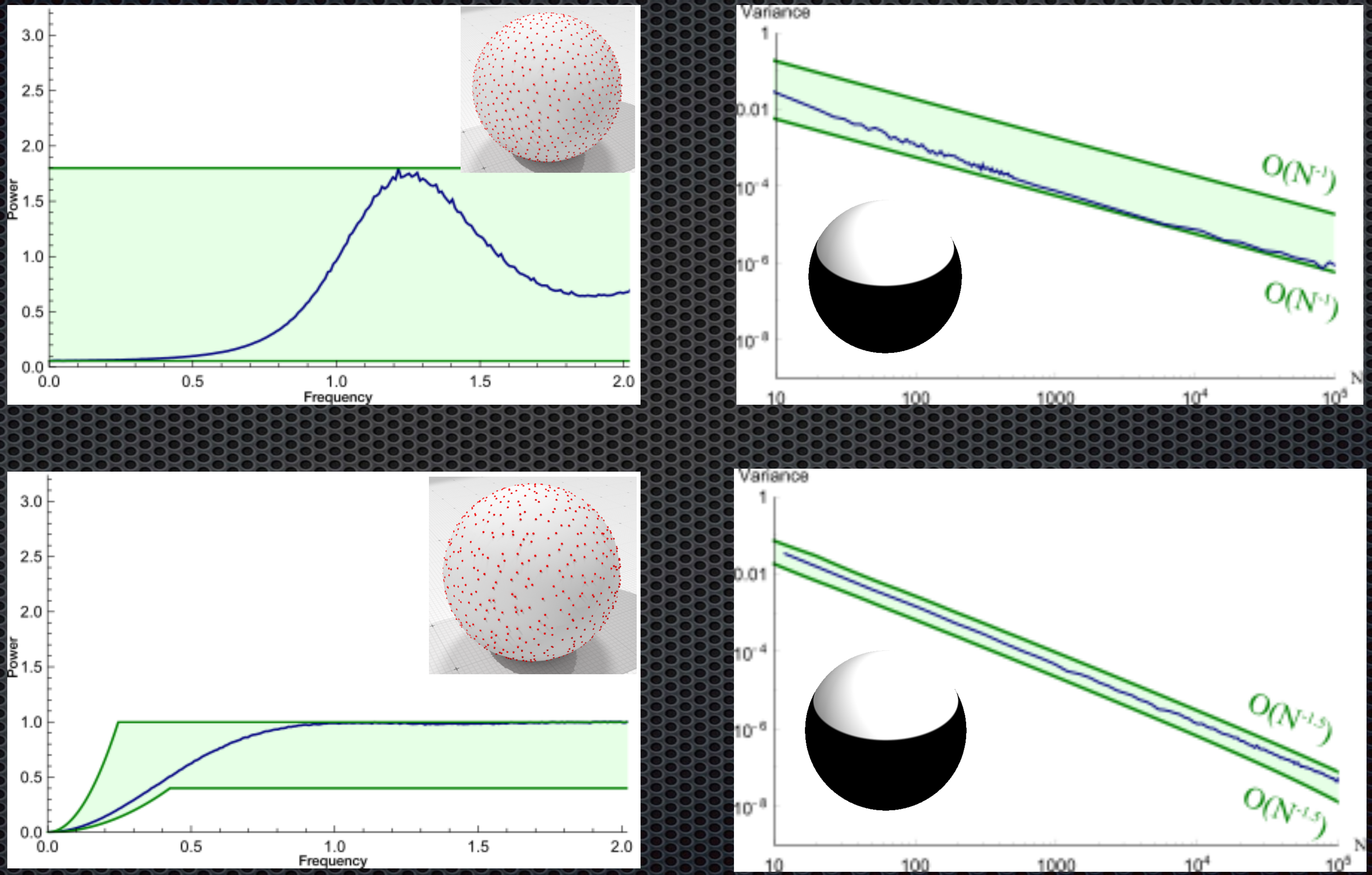
Power spectra behavior at low frequencies matters !



Convergence Rate Analysis

Poisson Disk

Jittered



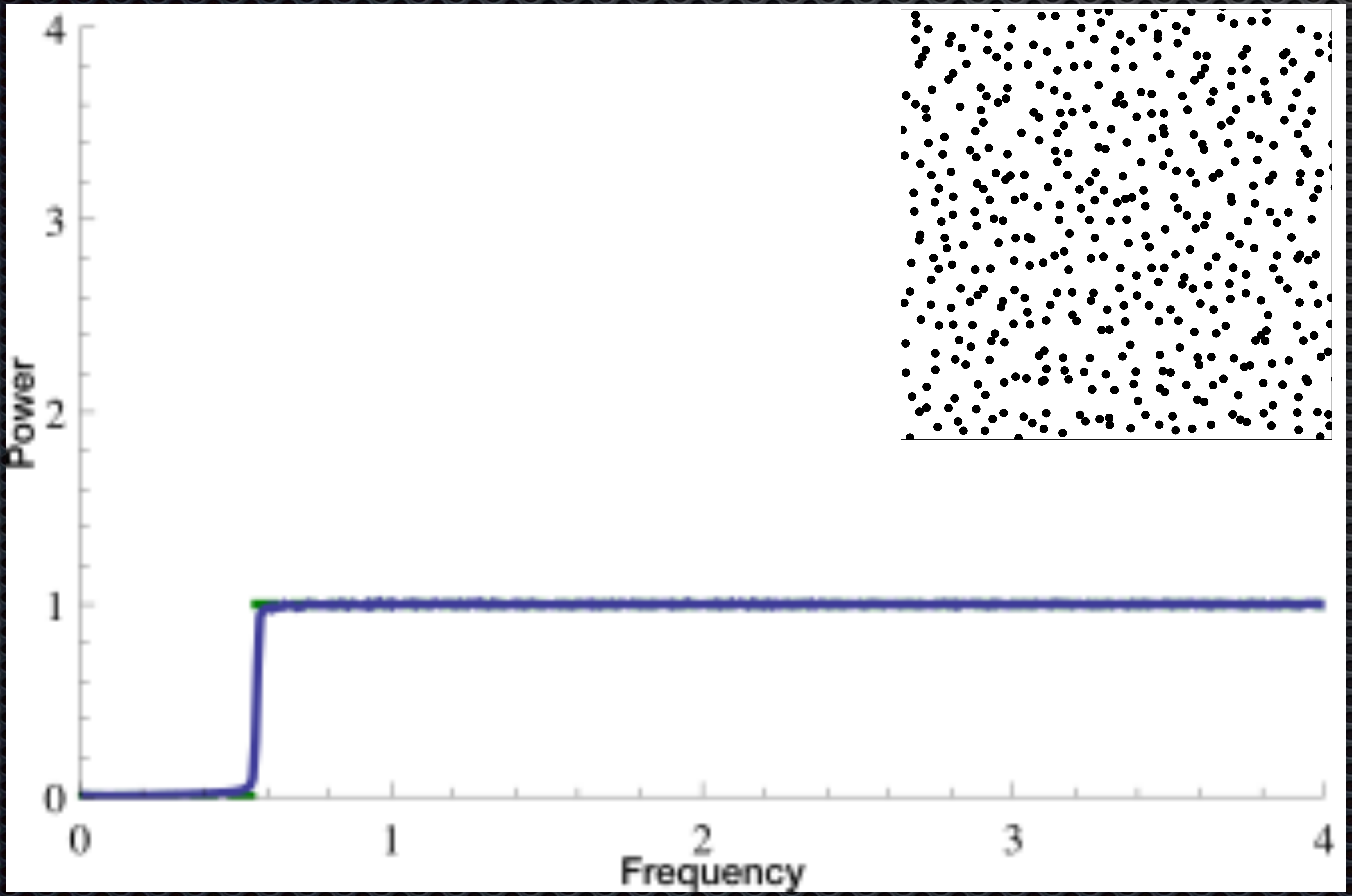
Power Spectrum

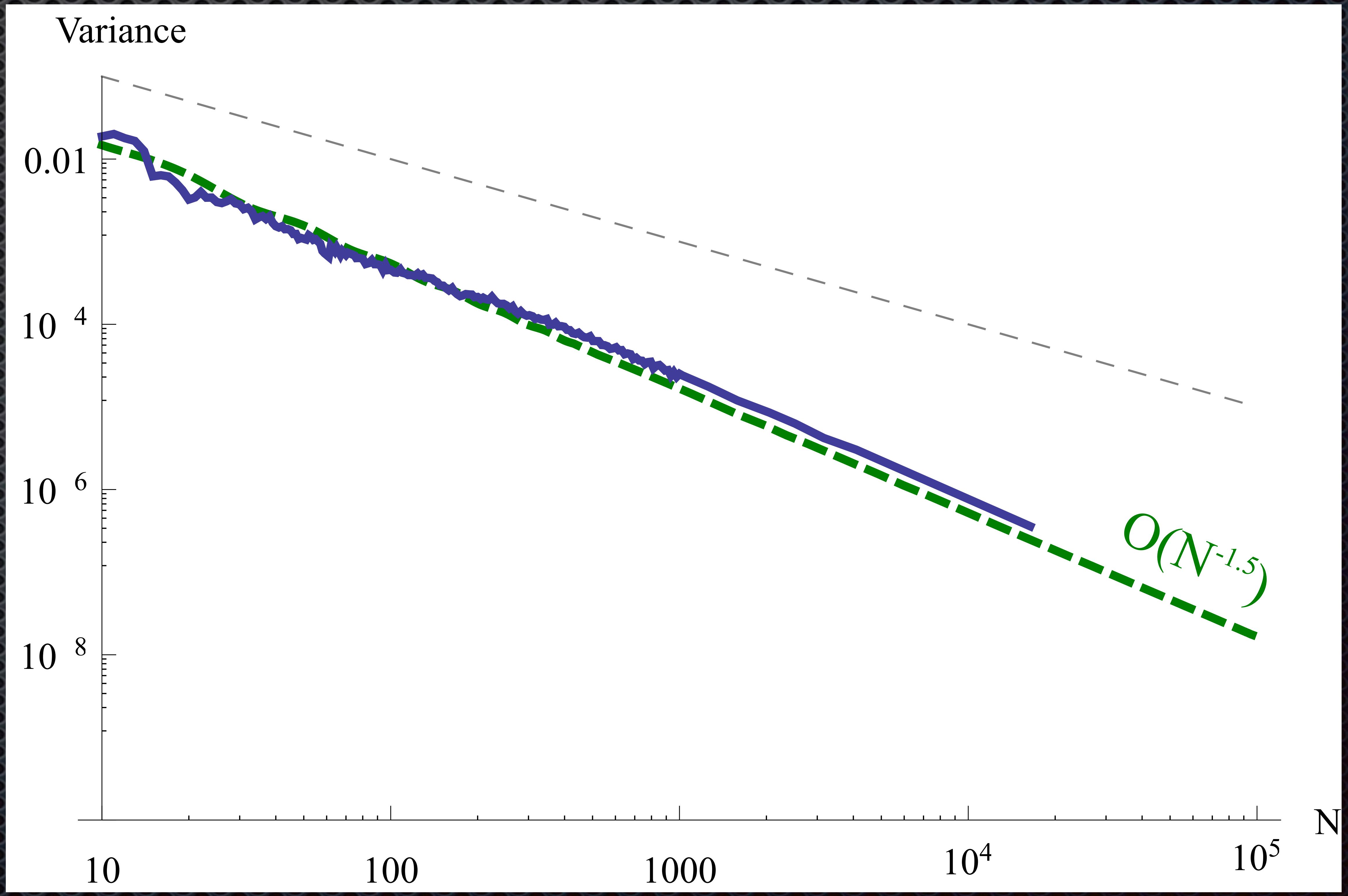
Convergence rate

$$O\left(\frac{1}{N}\right)$$

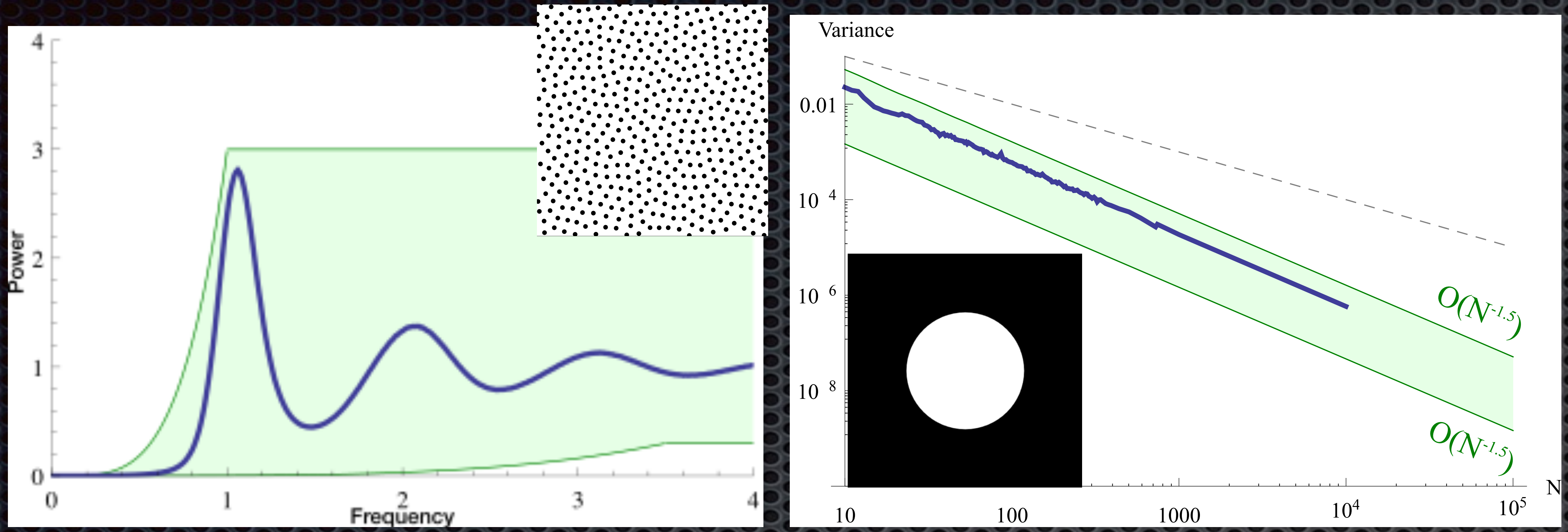
$$O\left(\frac{1}{N\sqrt{N}}\right)$$

Step Sampling Pattern

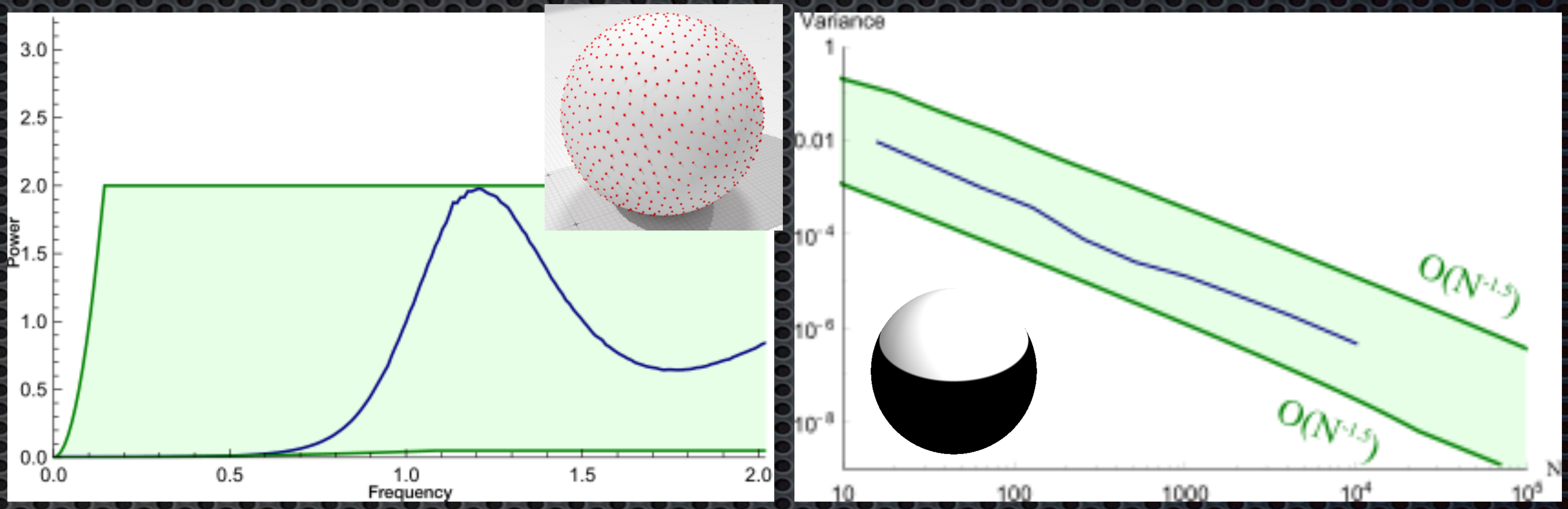




Convergence Rate Analysis



BNOT de Goes et al. 2012



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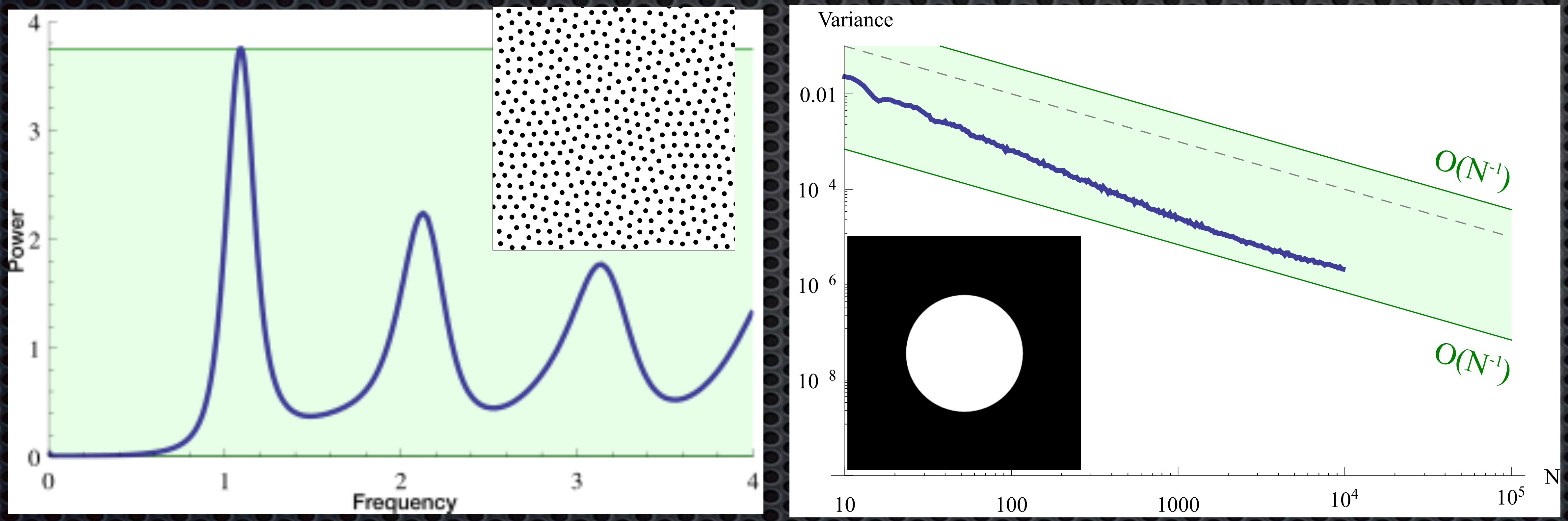
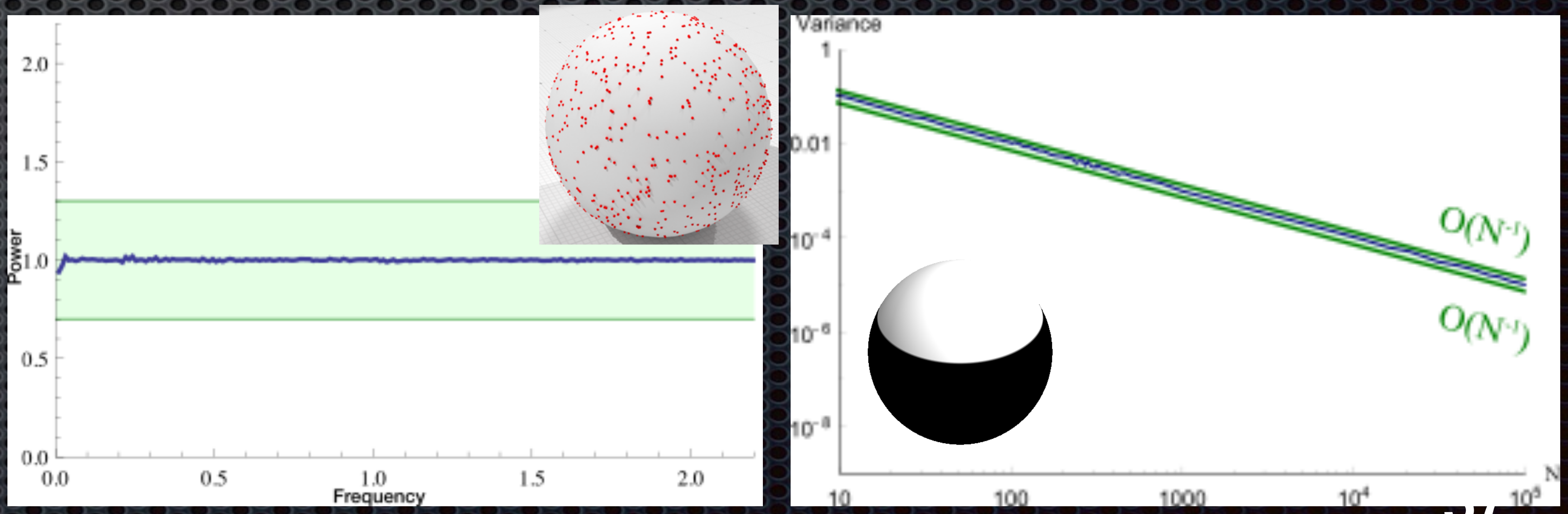
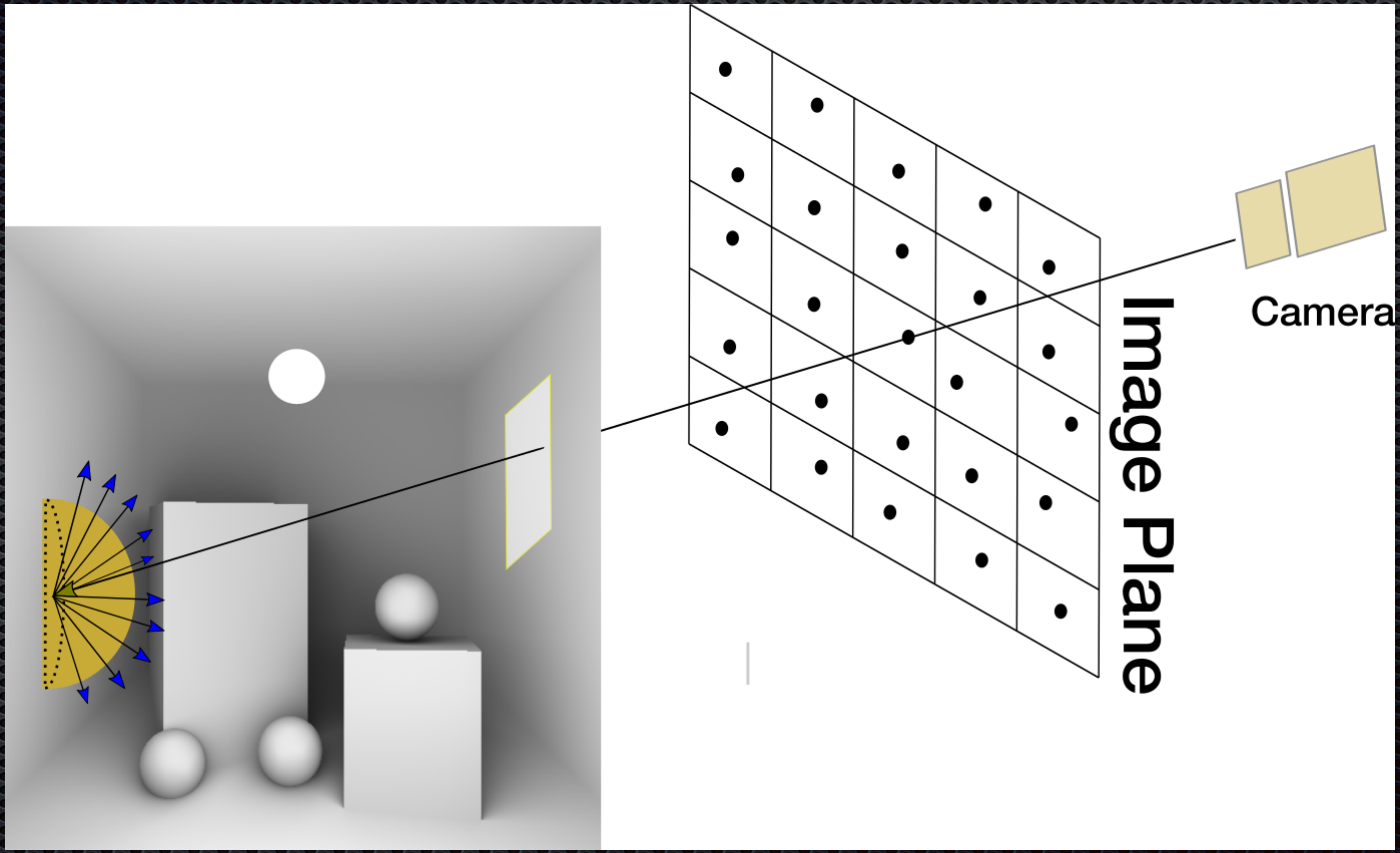


FPO Schlomer et al. 2011

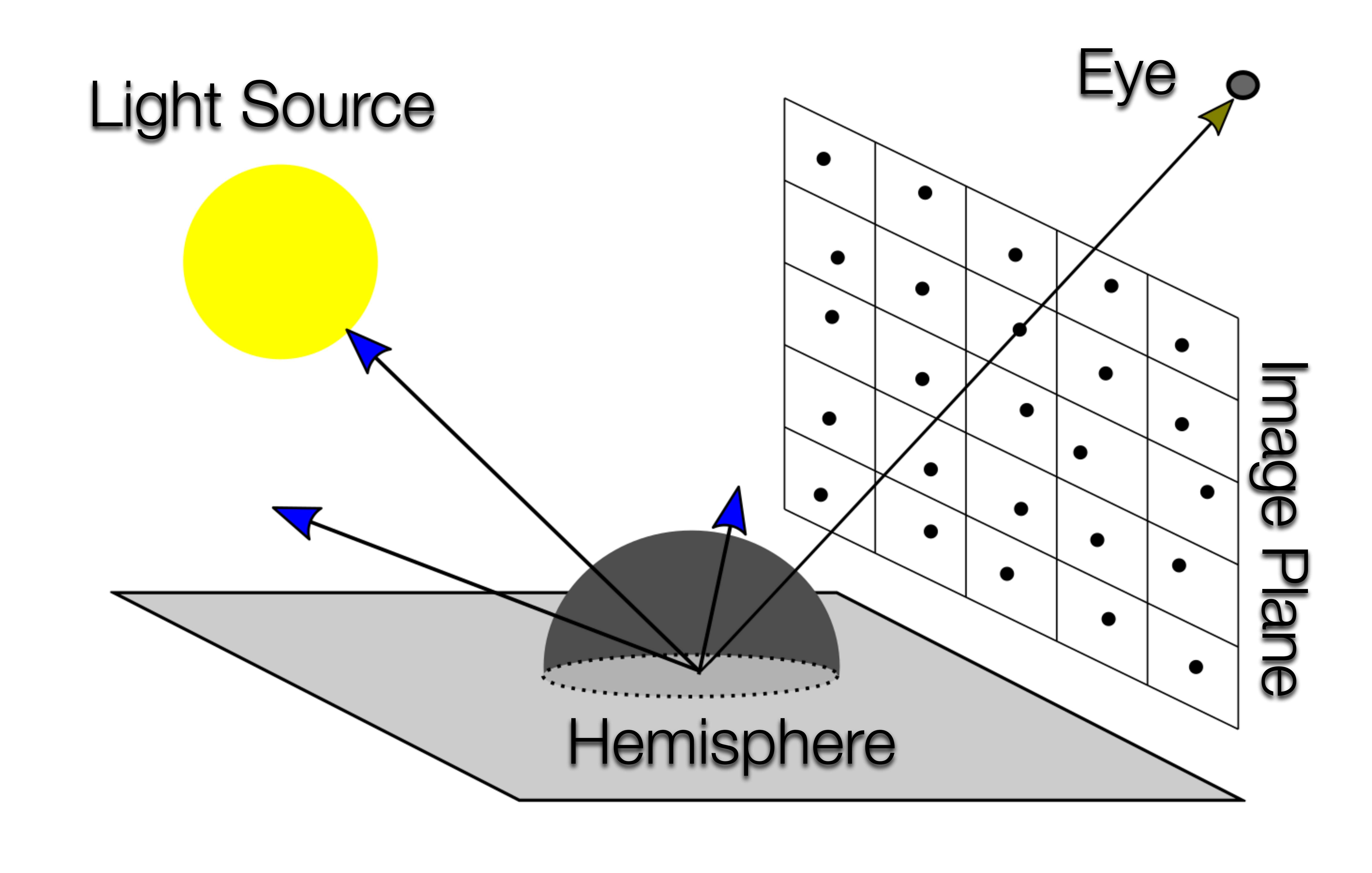


Purely random samples

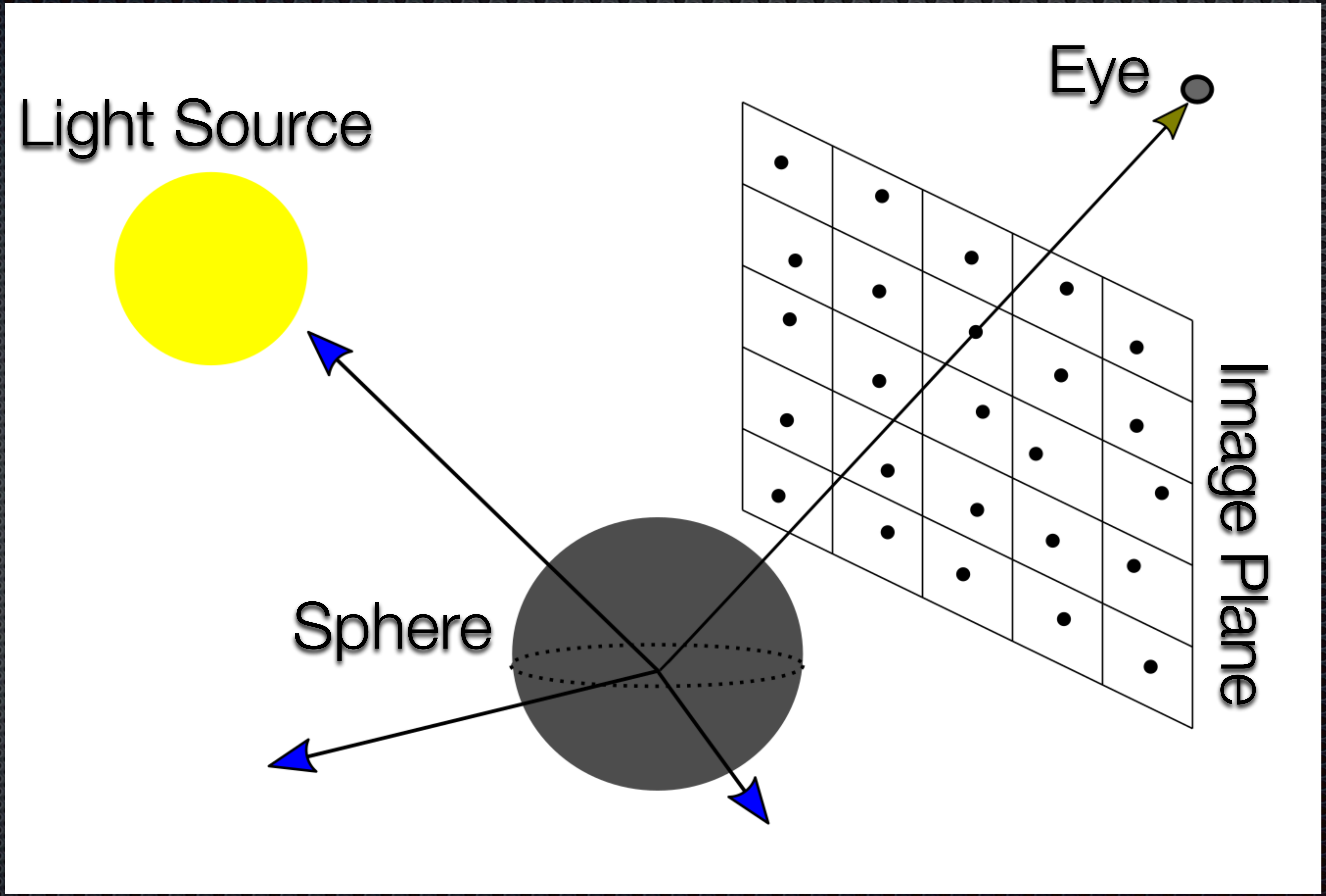
Light Simulation: on Surface



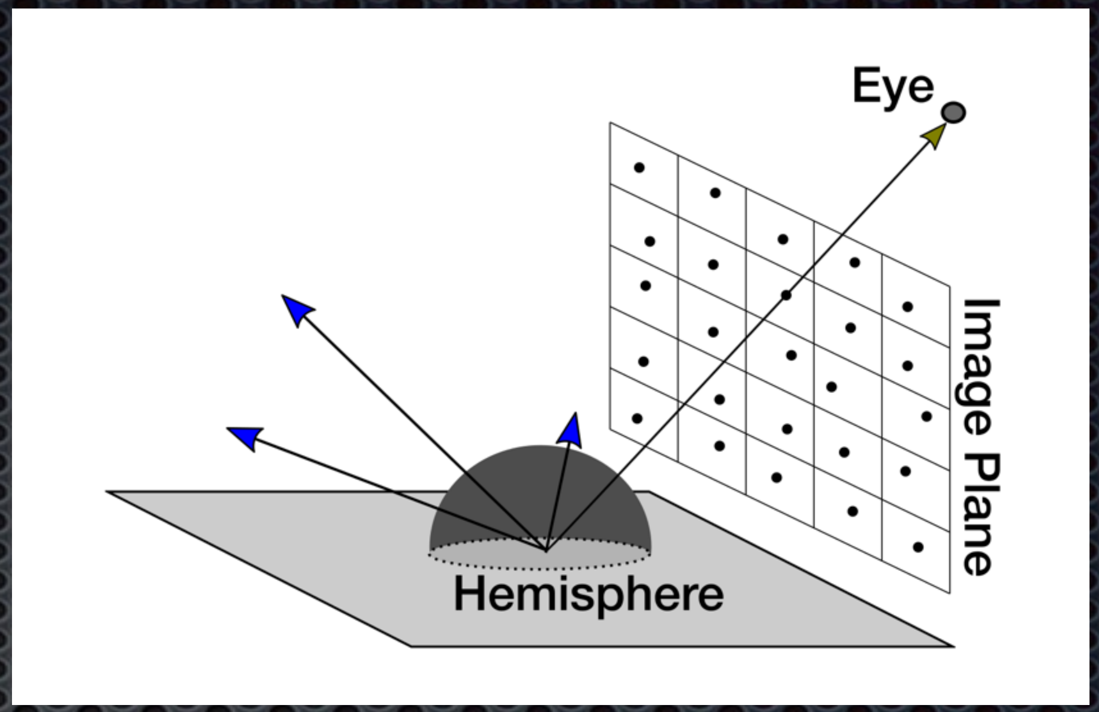
Light Simulation: on Surface

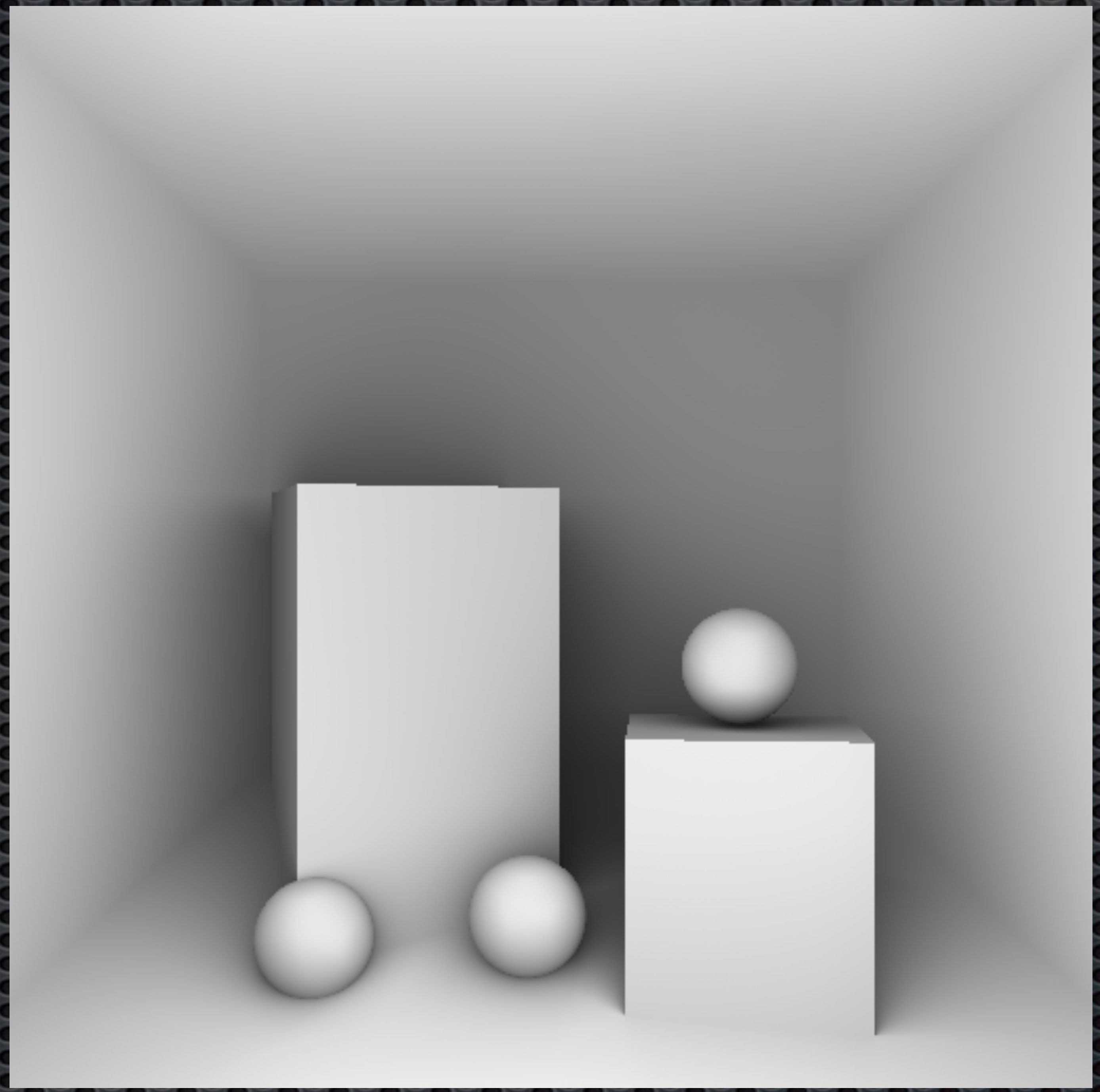


Light Simulation: Participating media



Ambient Occlusion





Structural Artifacts

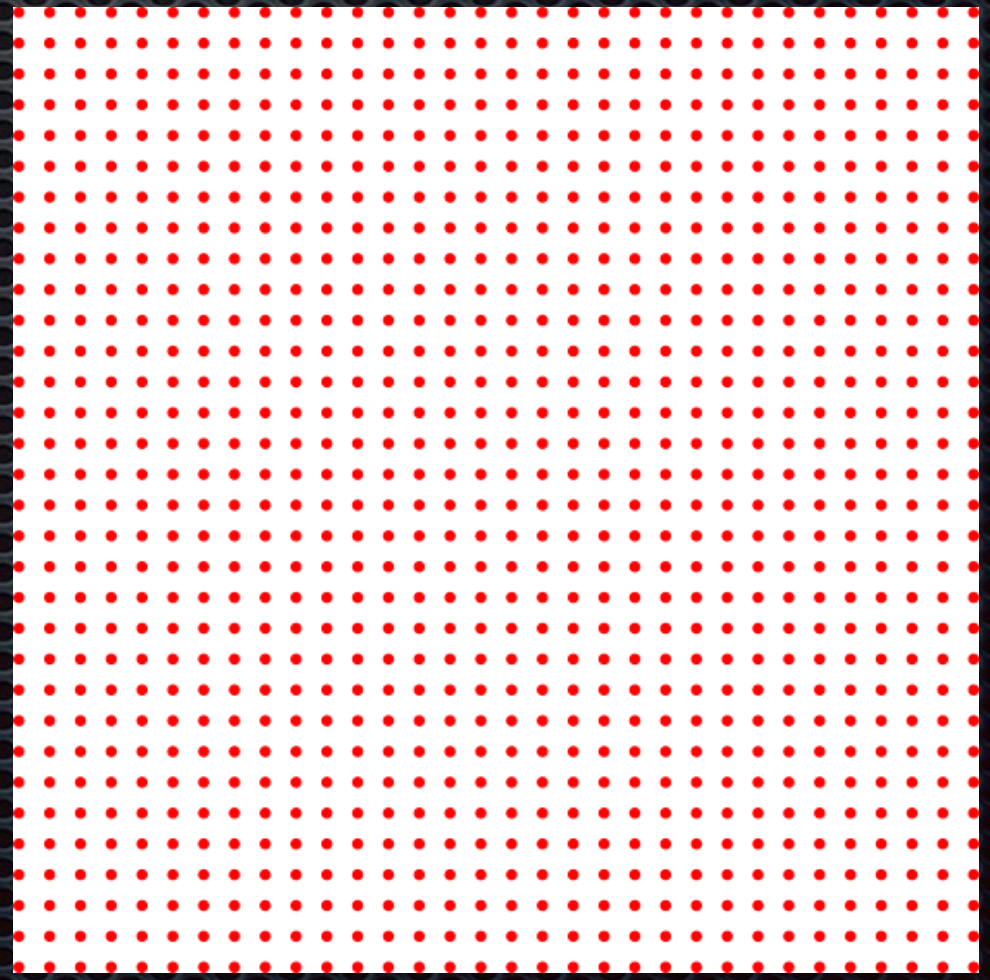
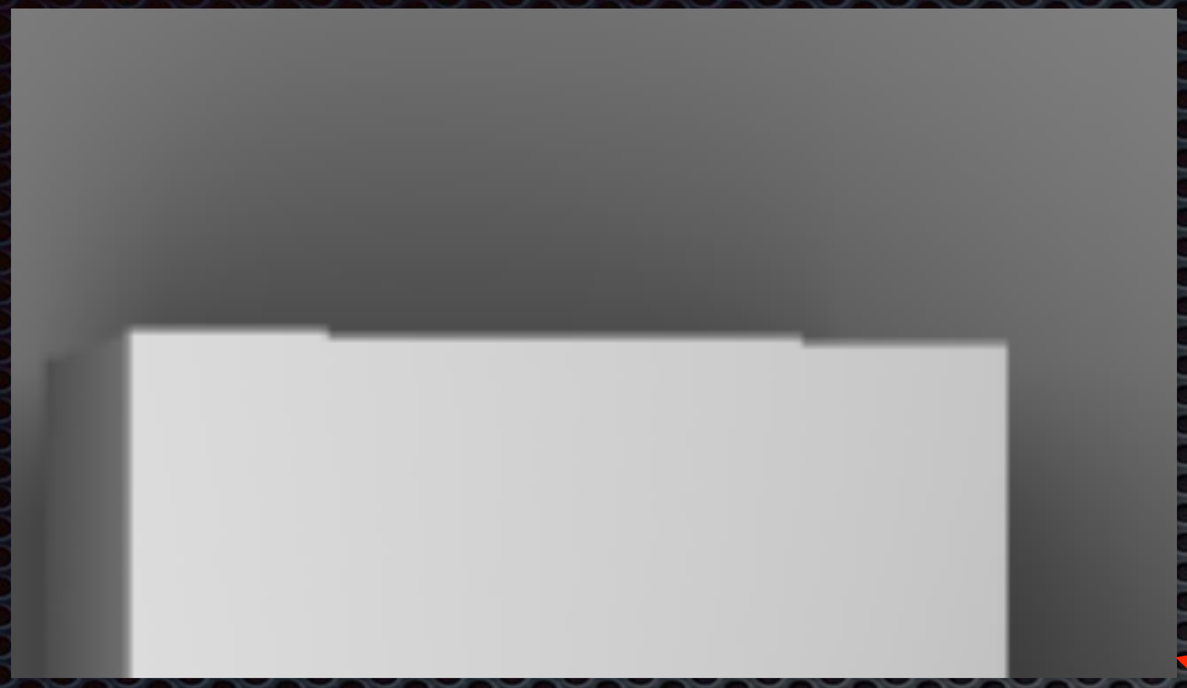
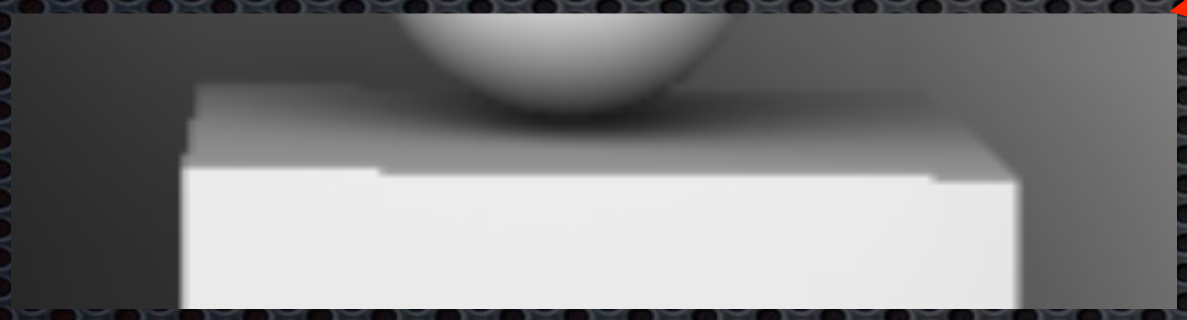
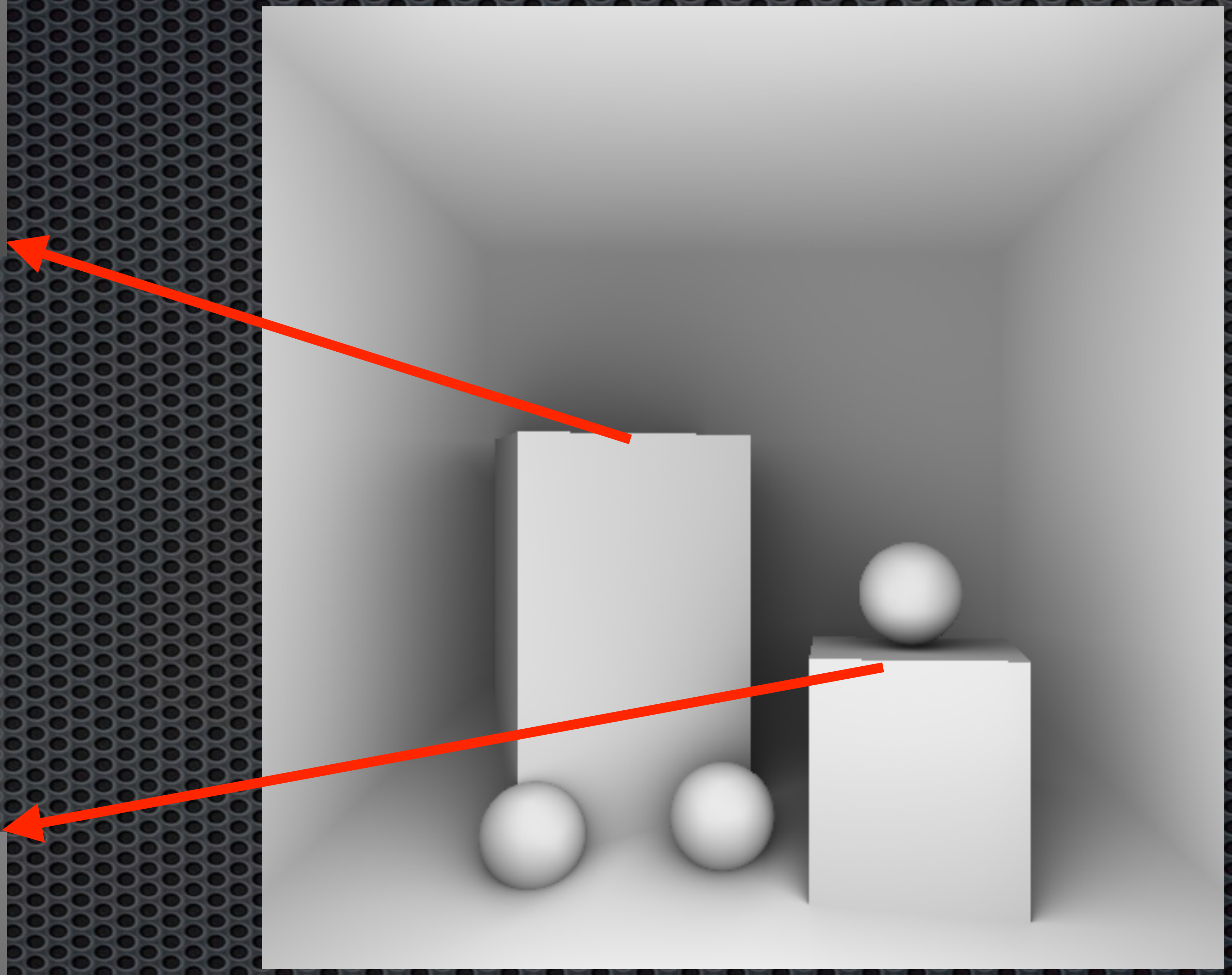


Image Plane

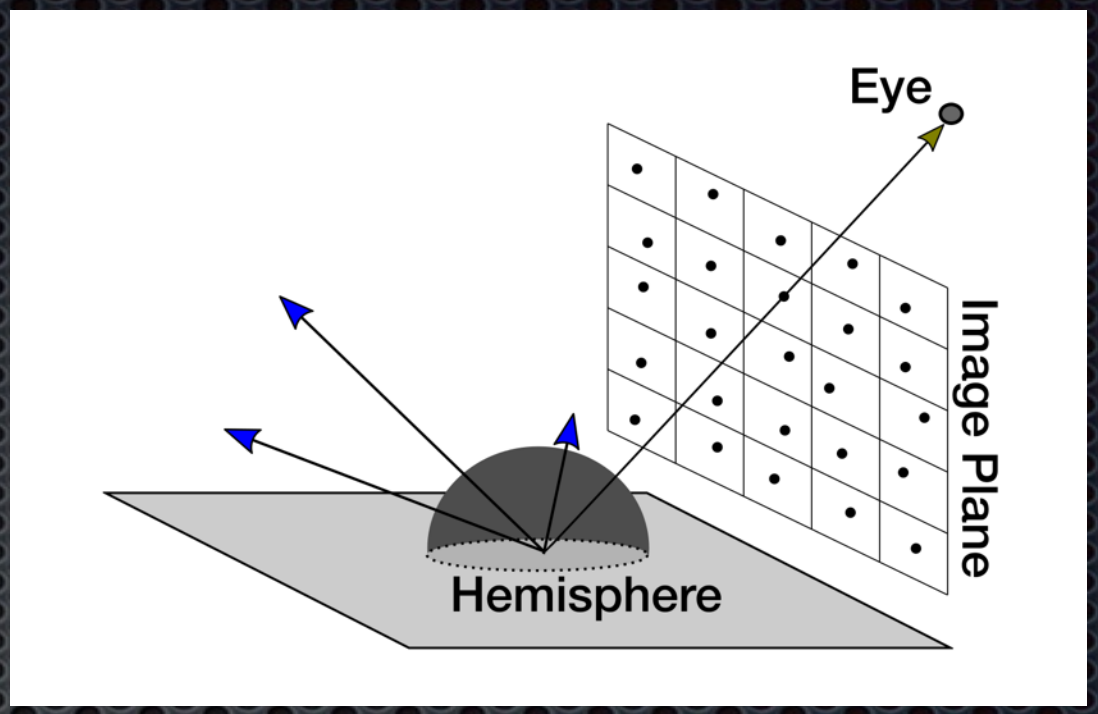
Ambient Occlusion







Structural Artifacts



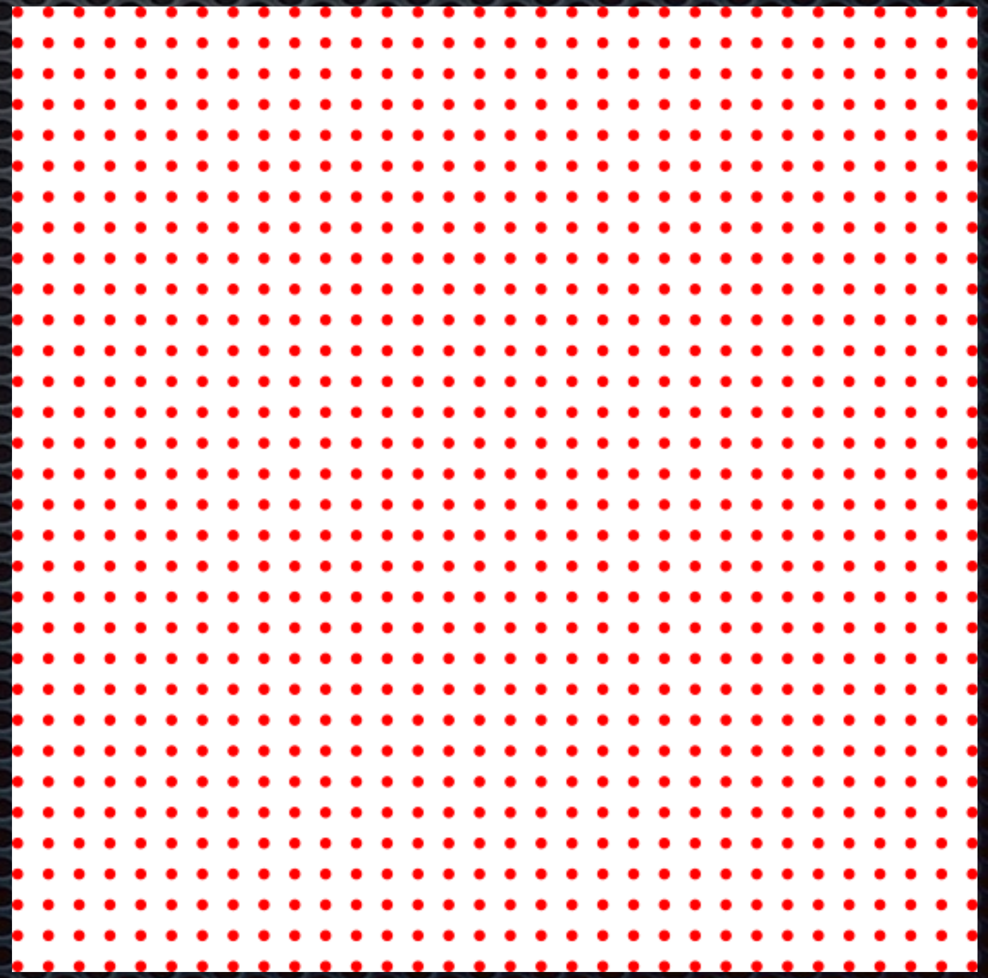
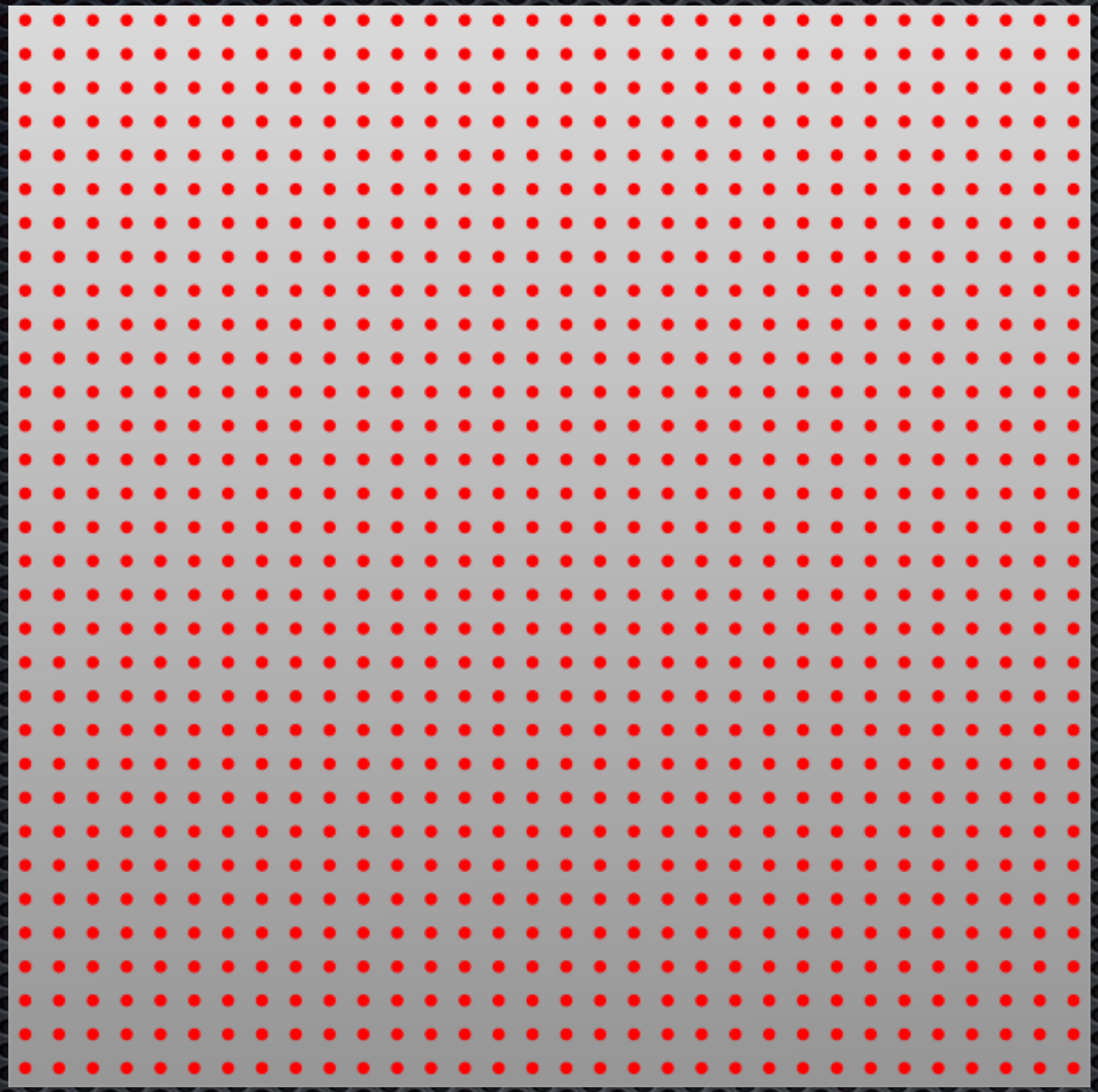
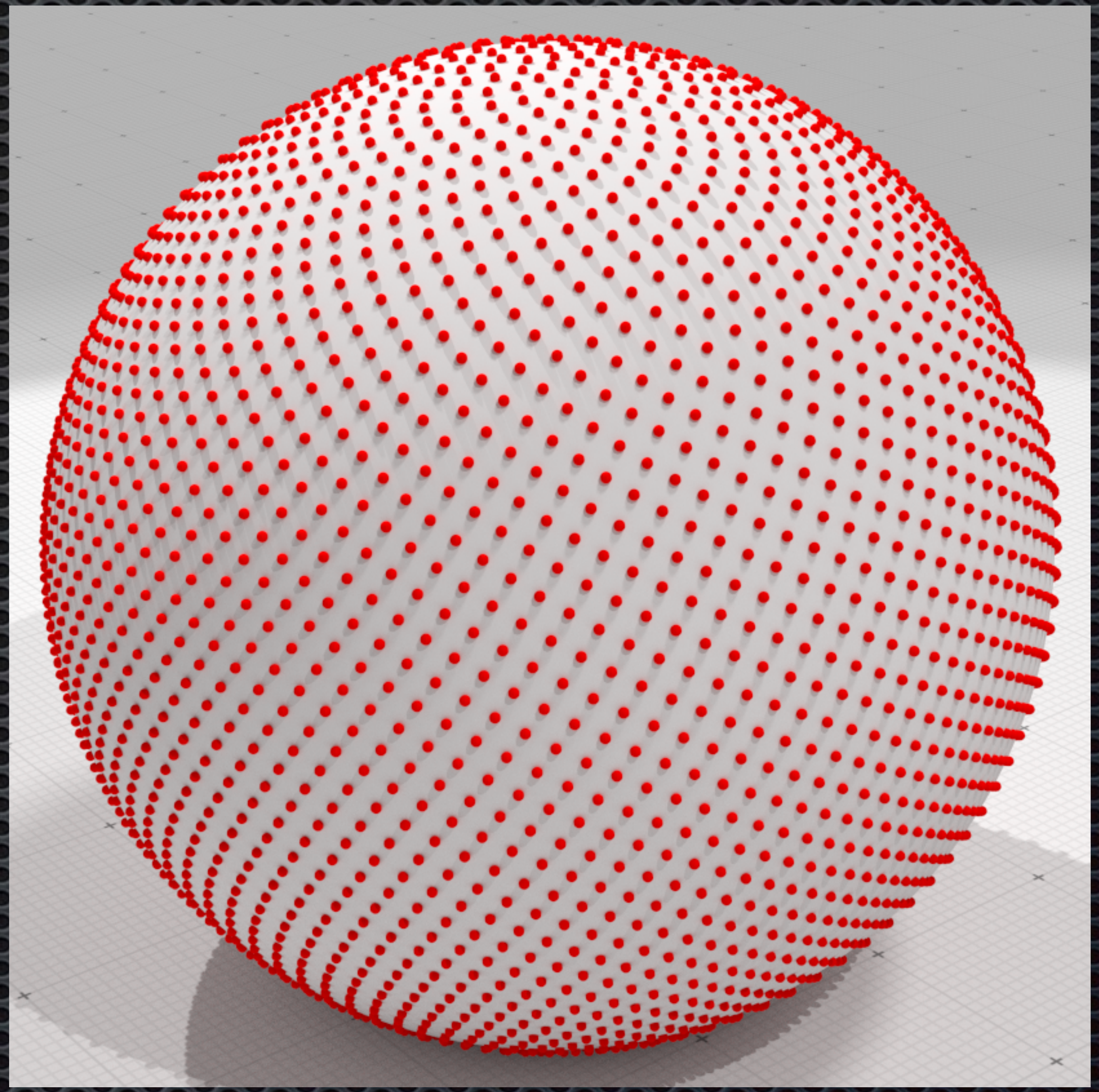


Image Plane

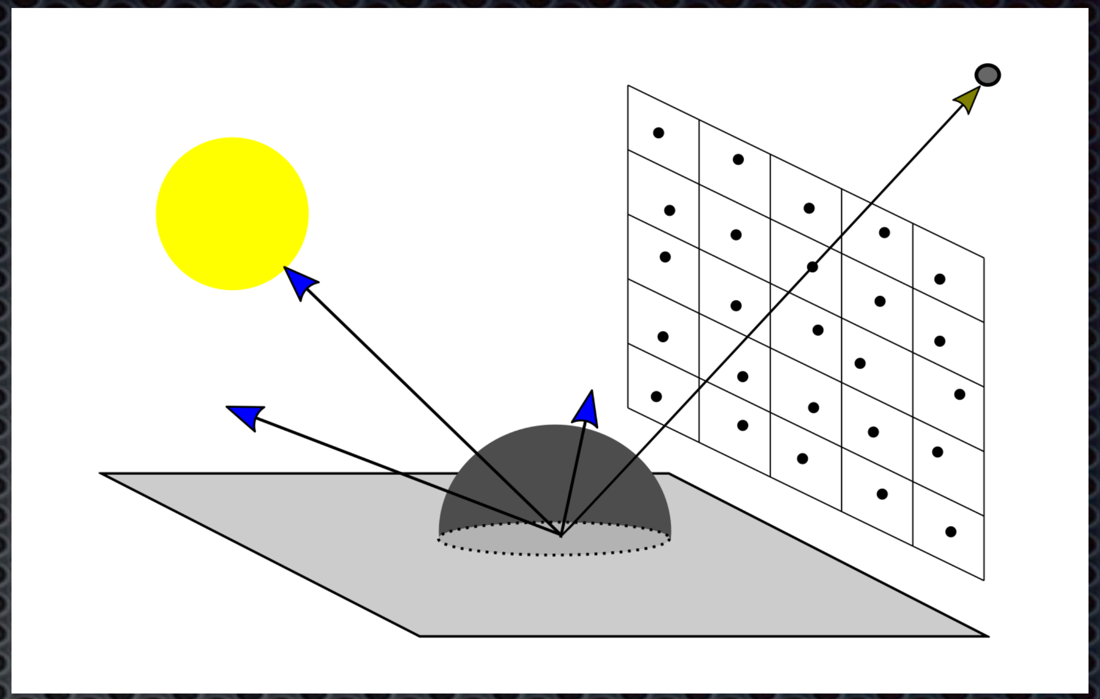
Euclidean 2D

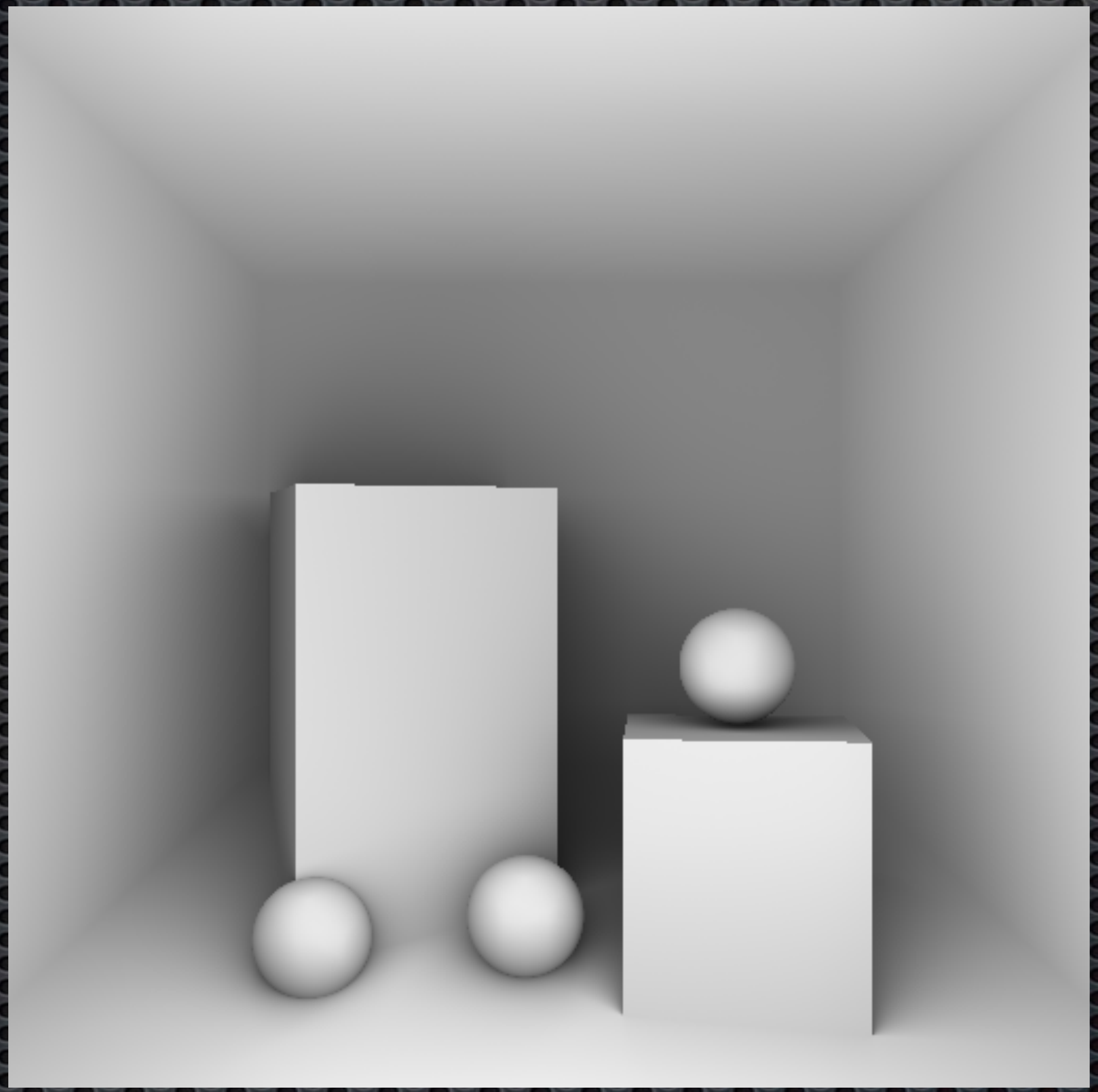


Spherical



Regular samples





Structural Artifacts

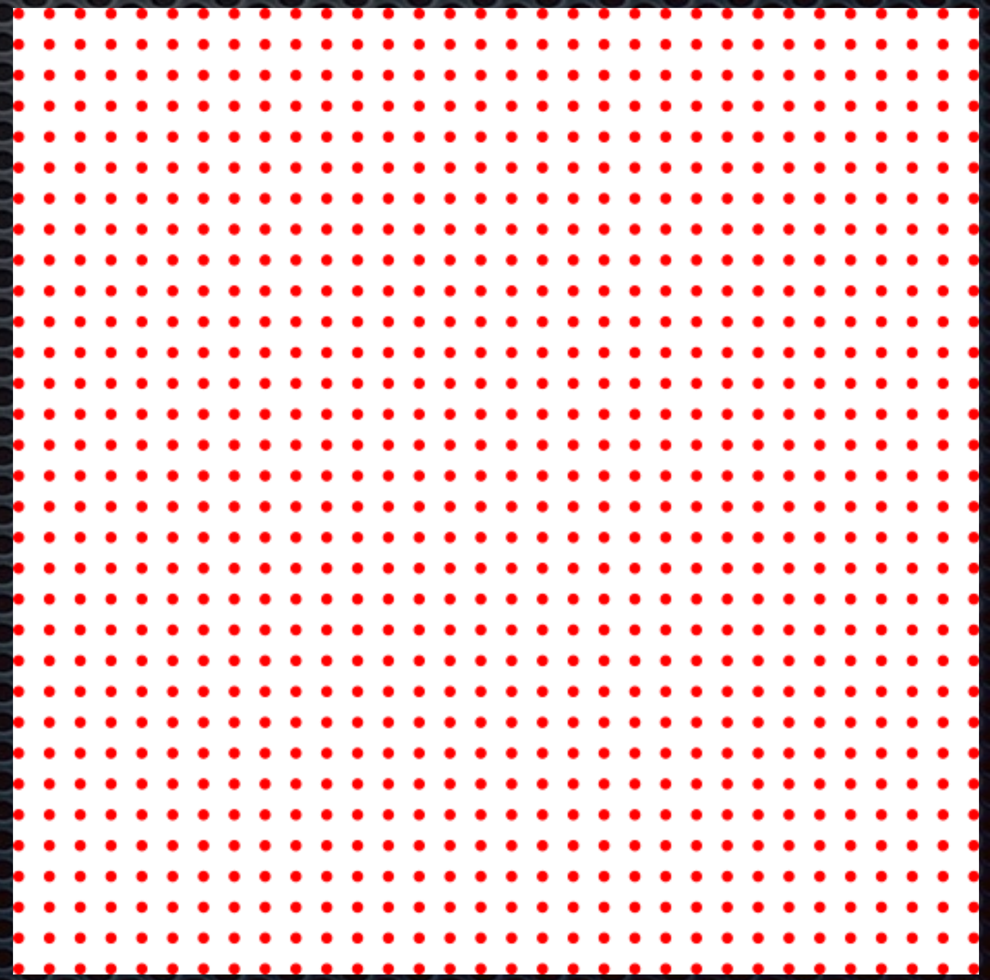
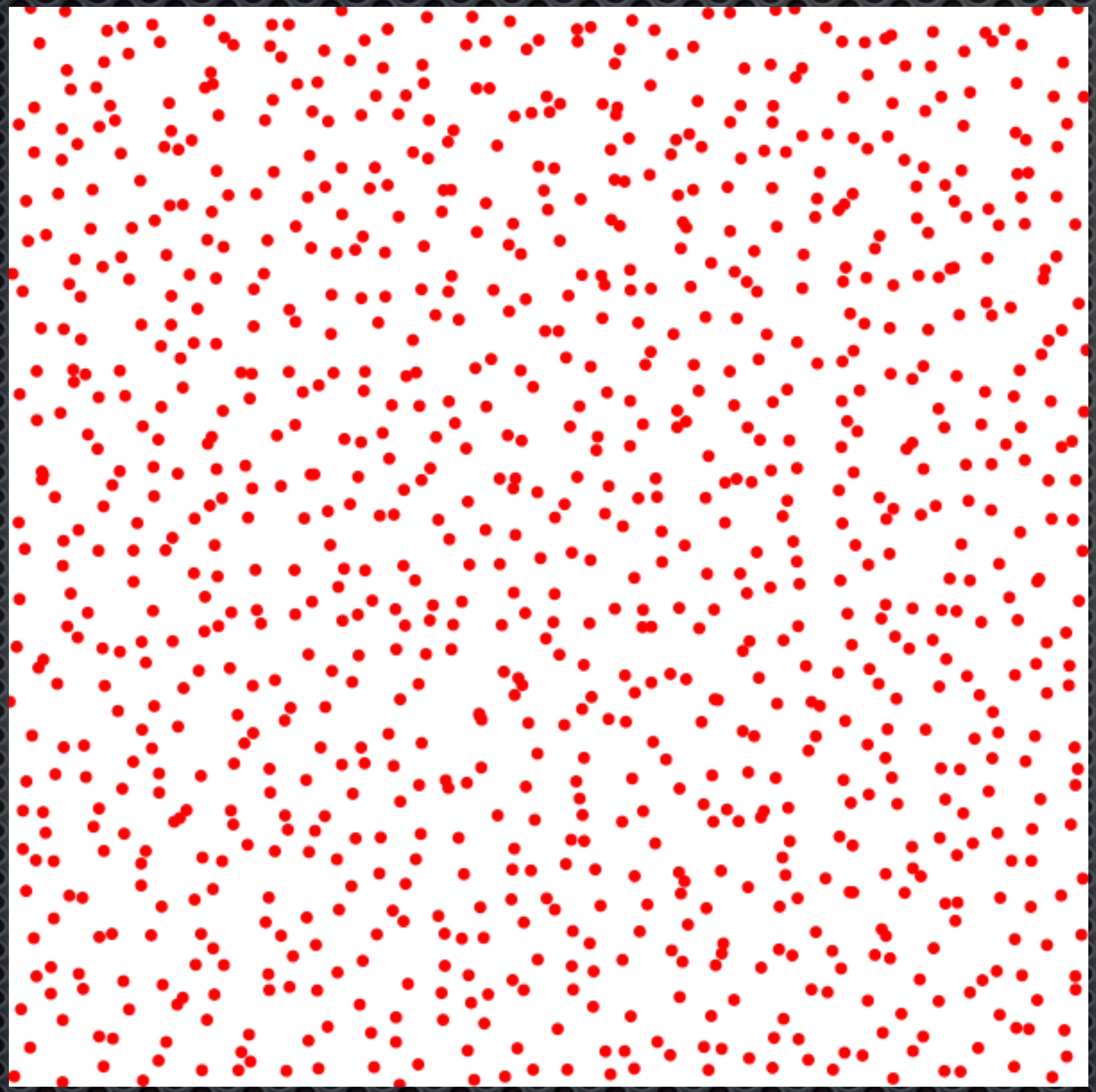


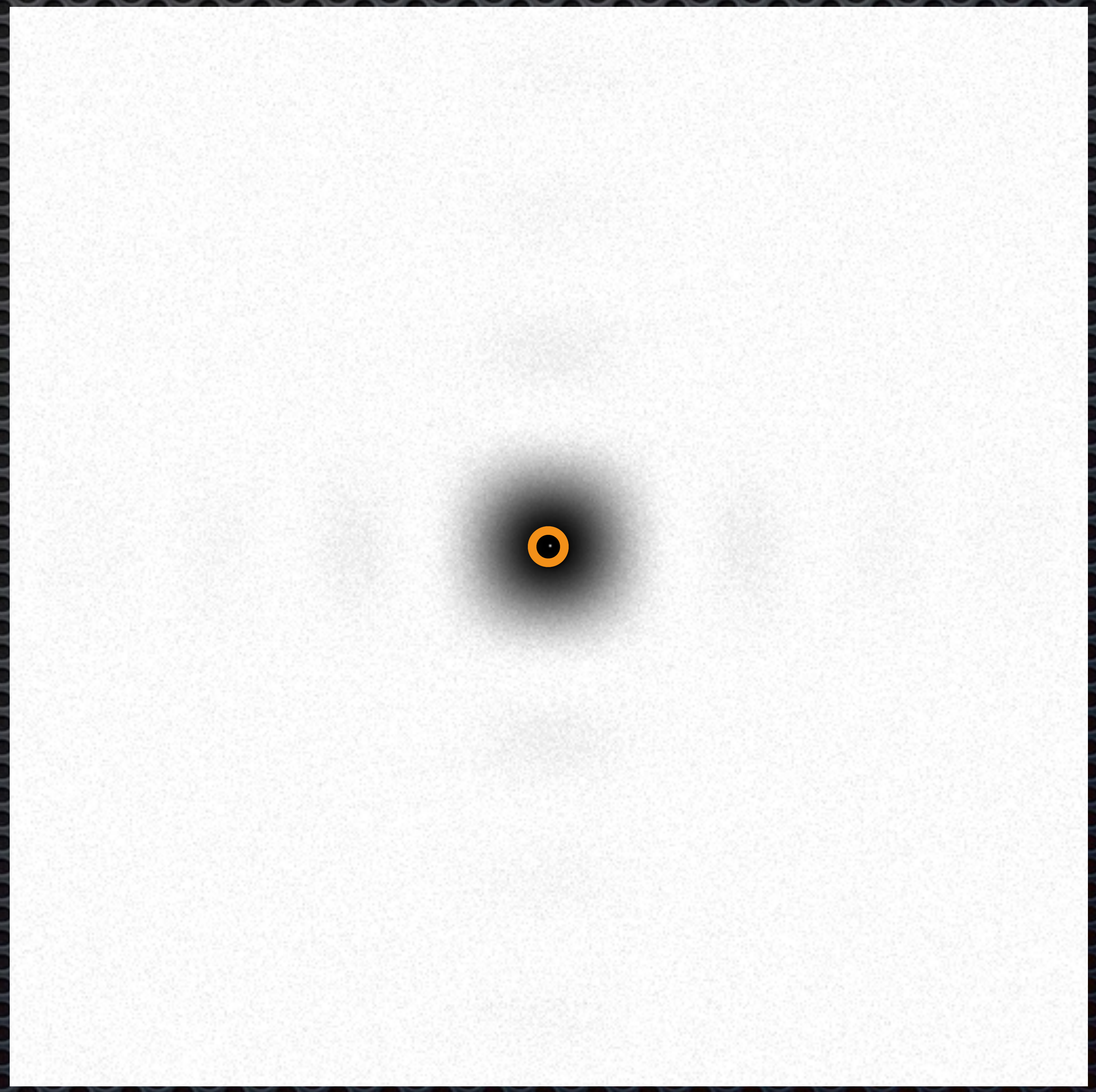
Image Plane

Jittered Sampling Pattern

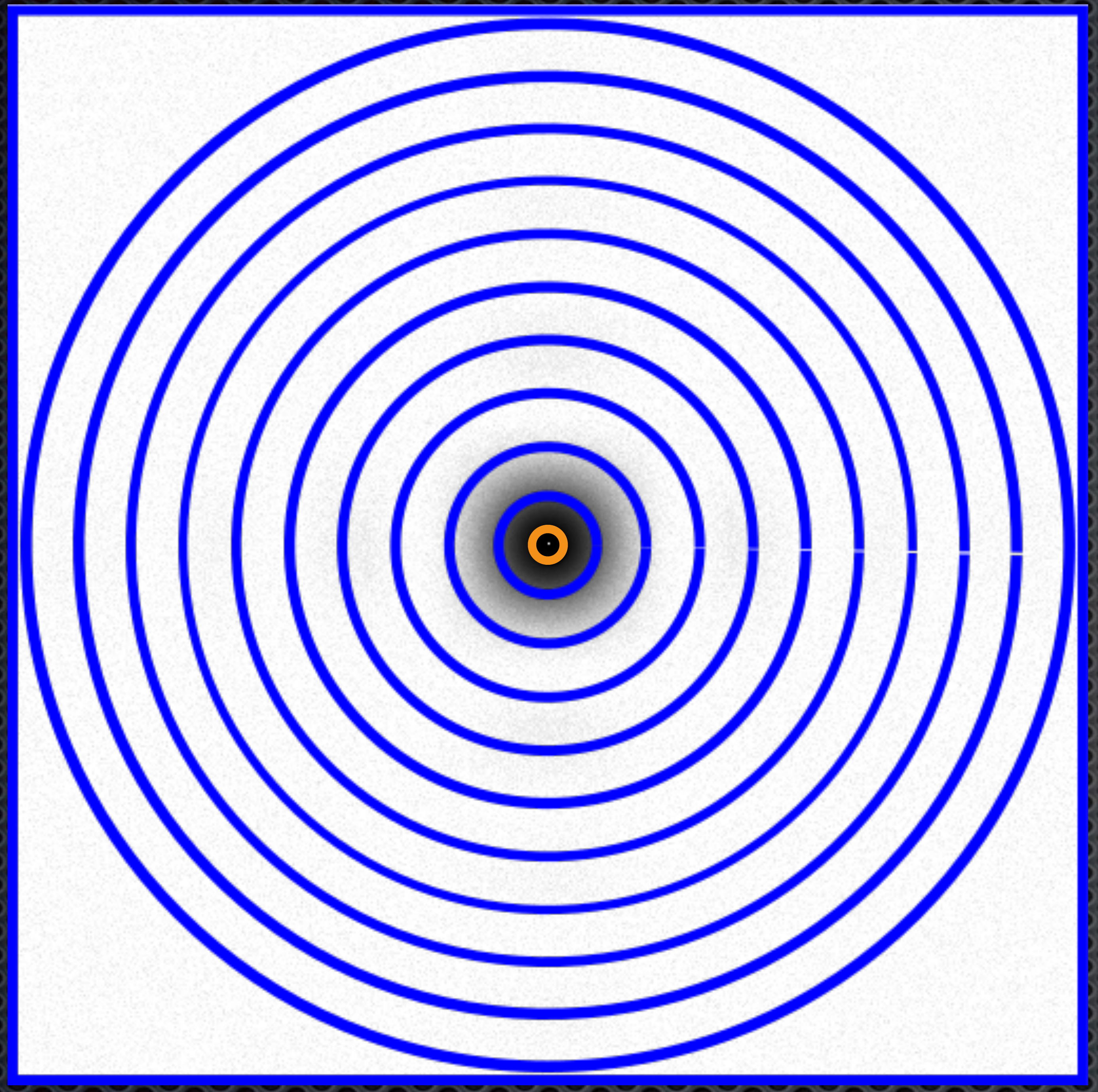
Samples

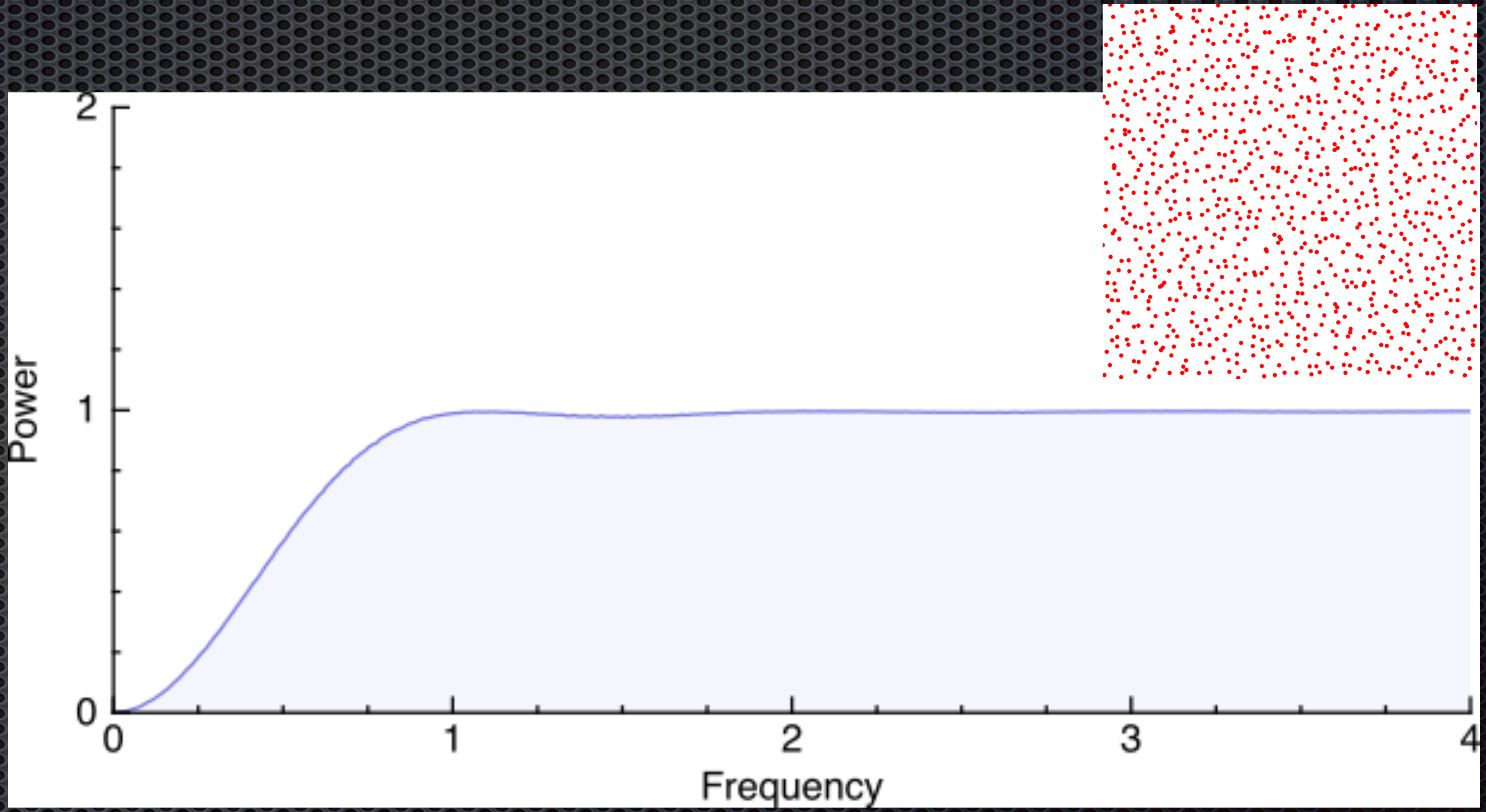


Power Spectrum



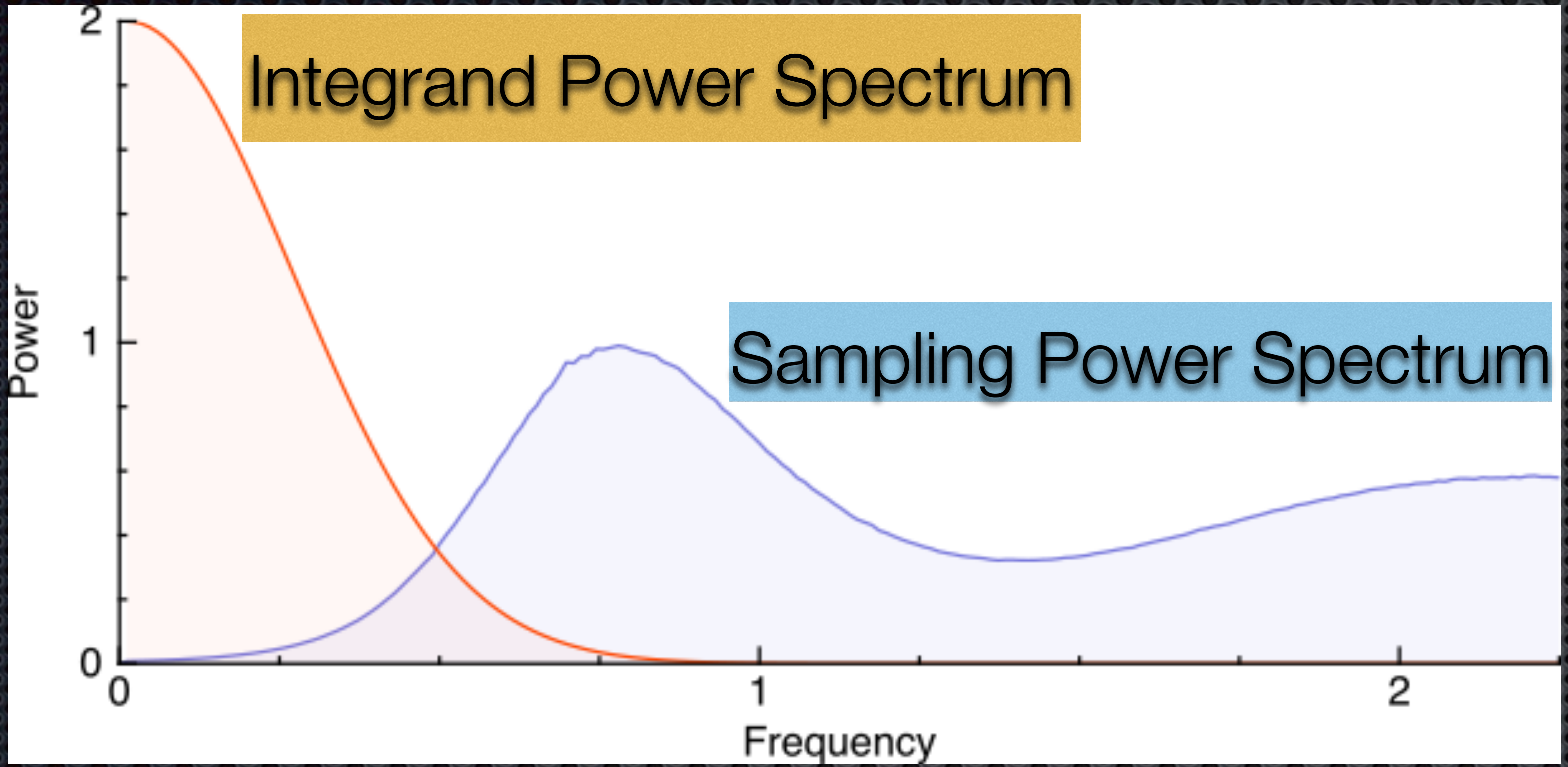
Radial averging of Power Spectrum





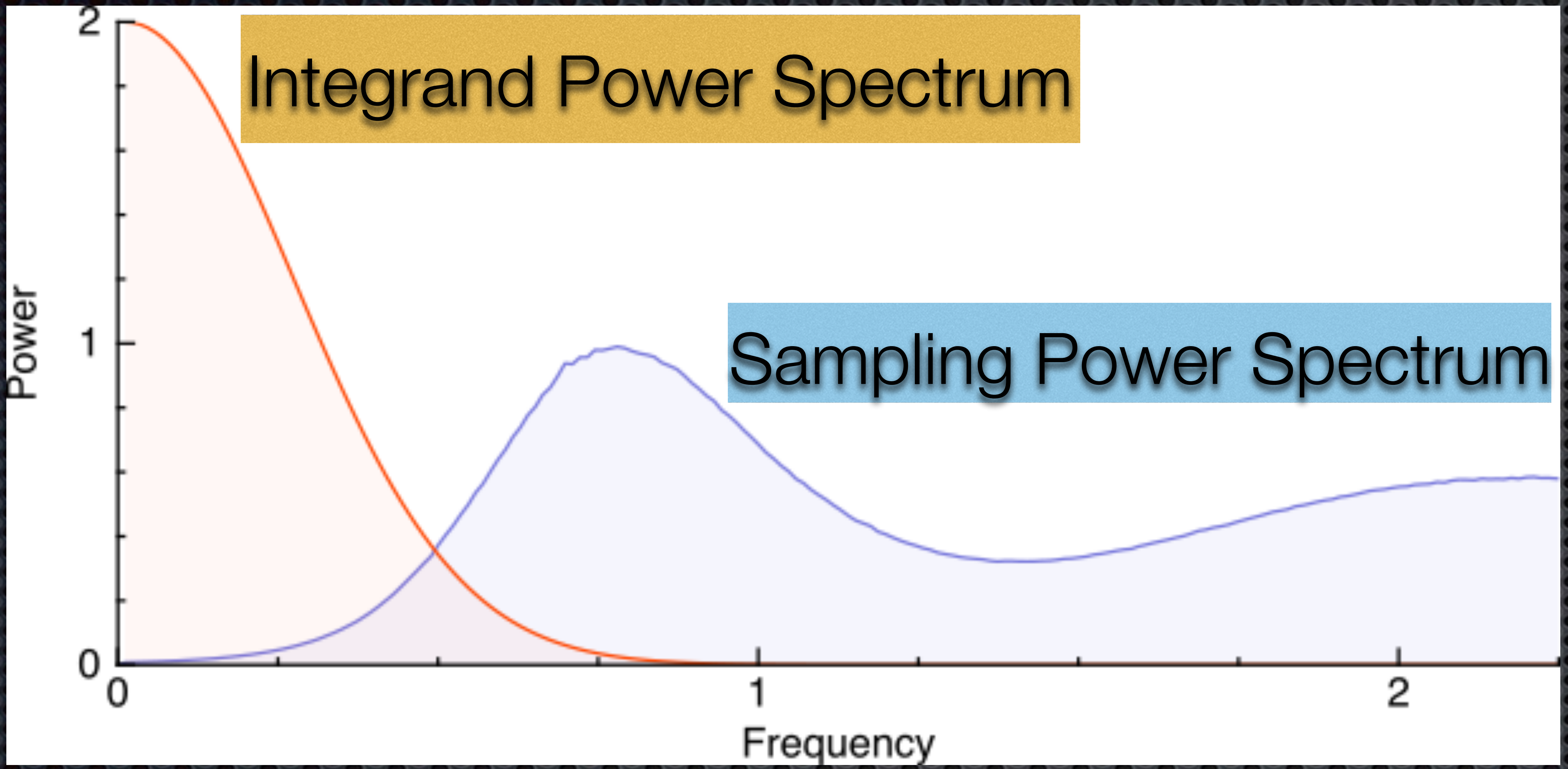
Jittered sampling Pattern

Dependence on Number of Samples

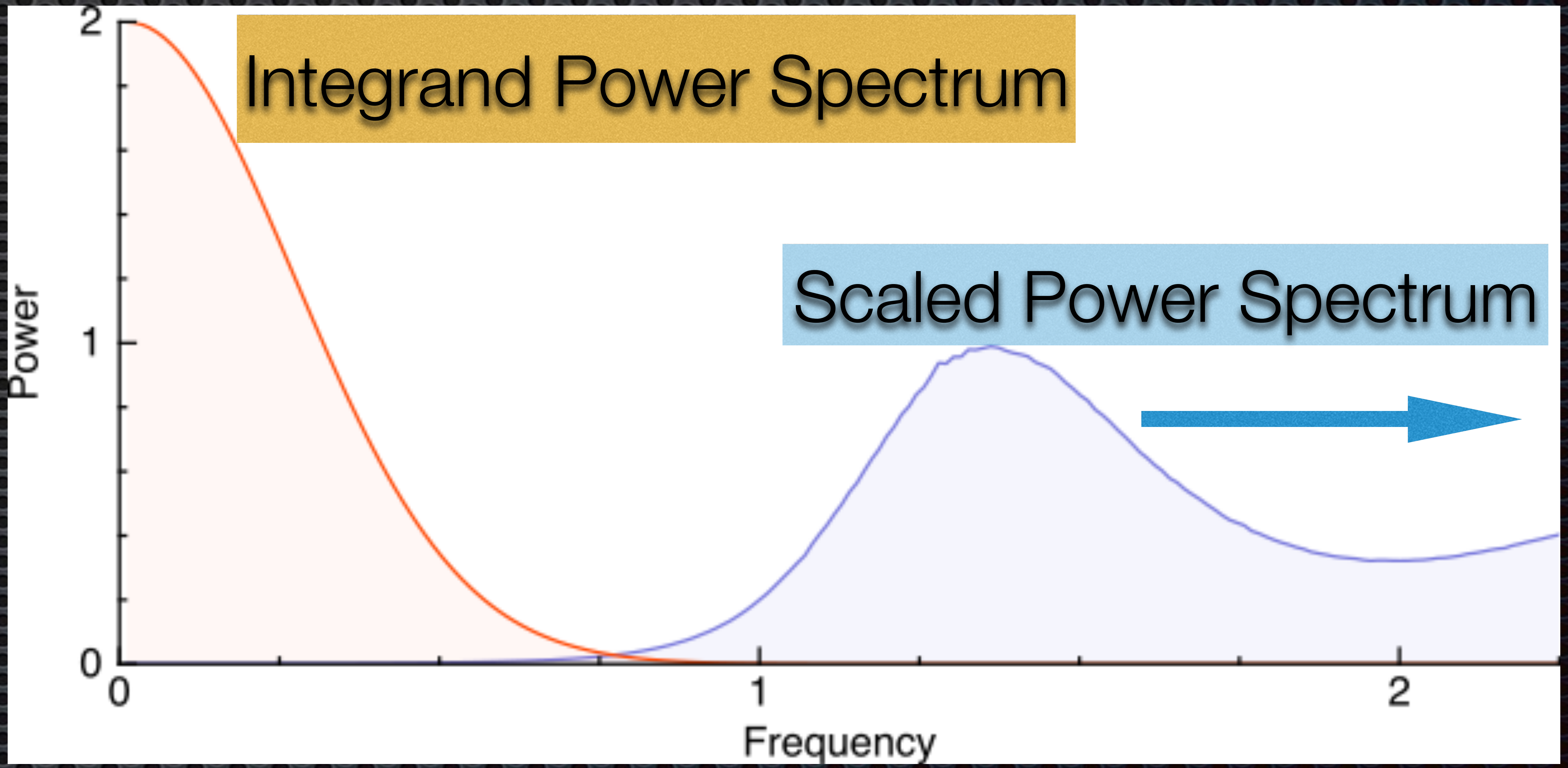


For a given number of samples

Dependence on Number of Samples

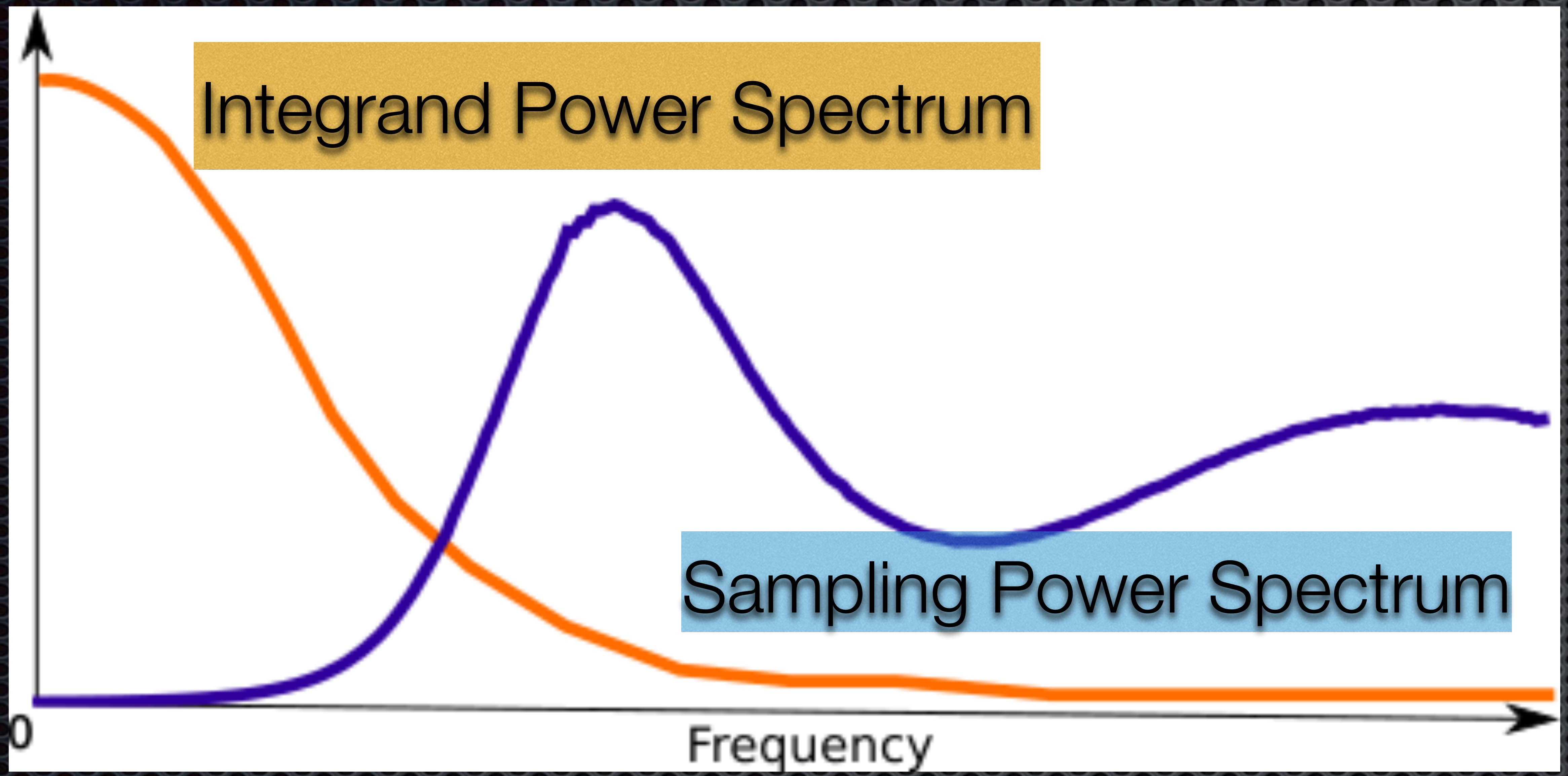


For a given number of samples



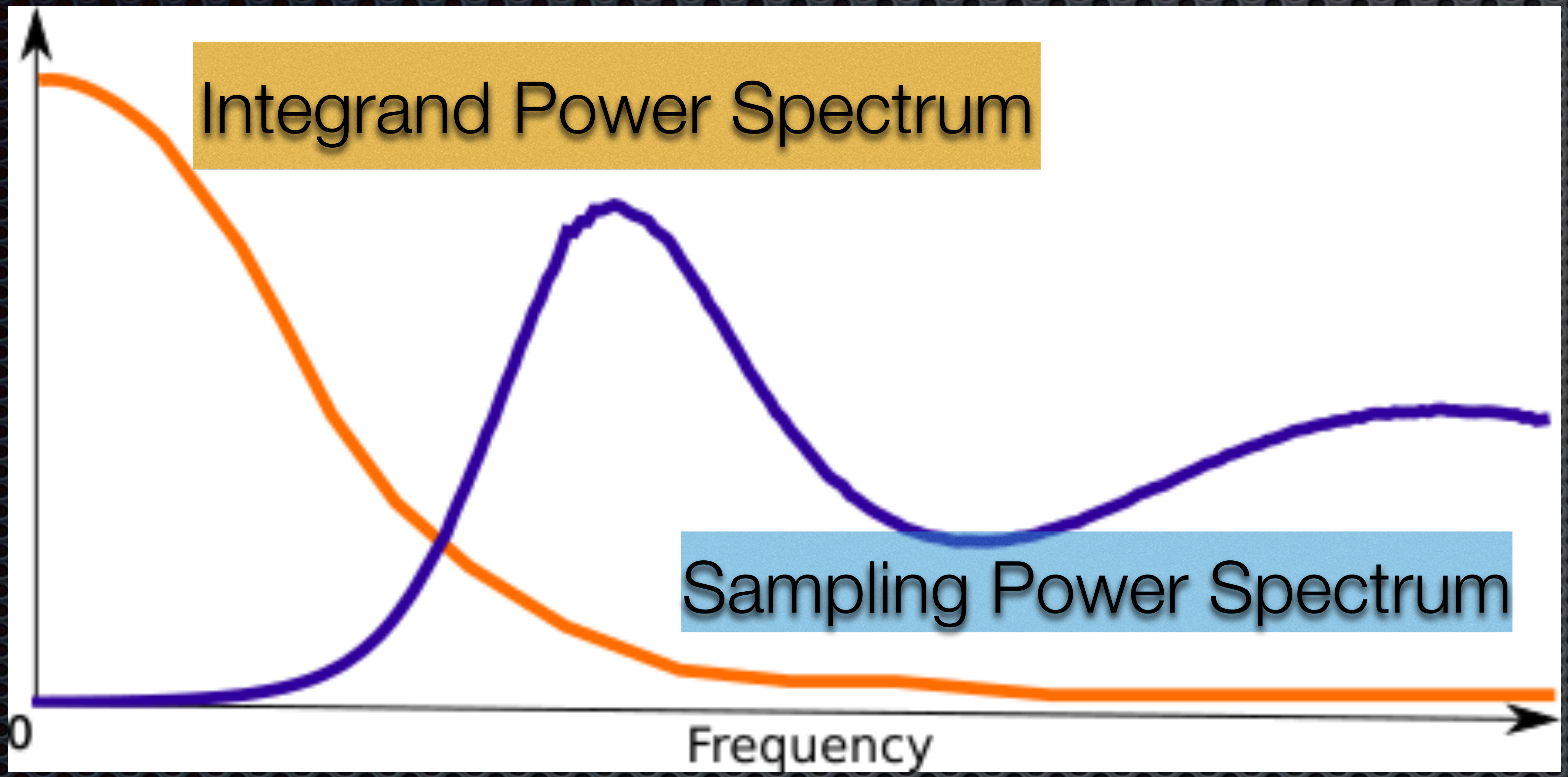
Increase in number of samples

Dependence on Number of Samples

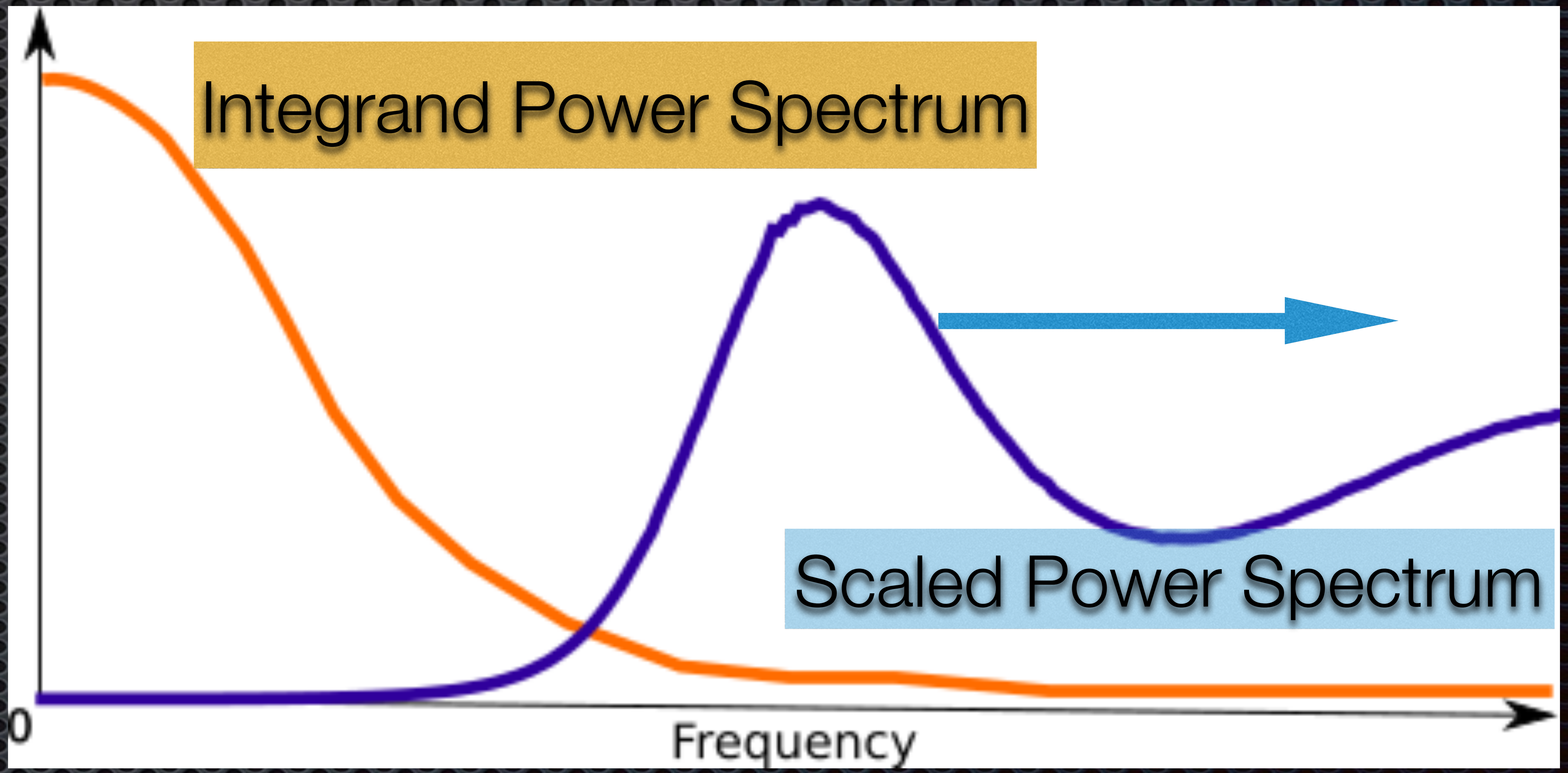


For a given number of samples

Dependence on Number of Samples

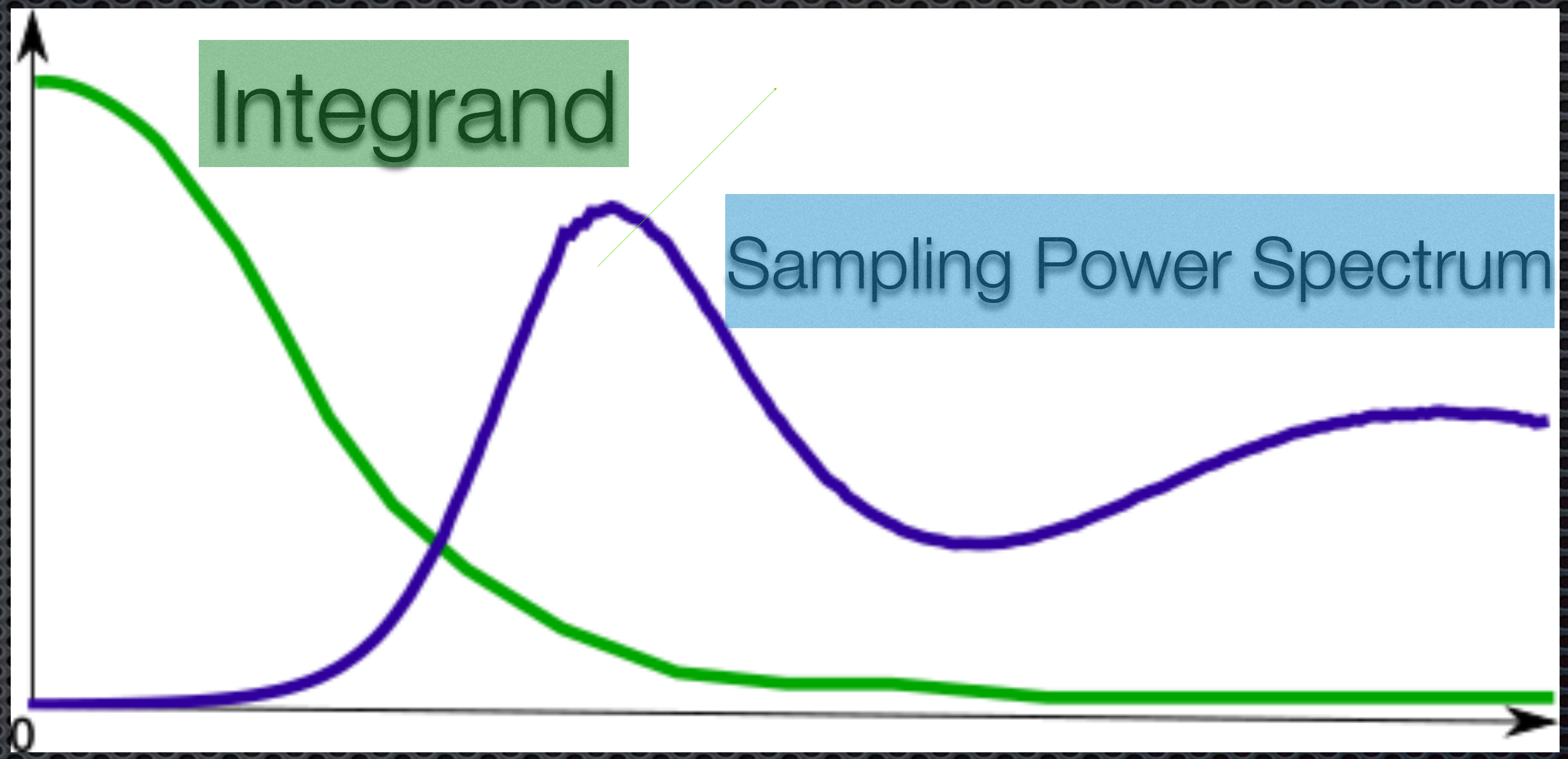


For a given number of samples

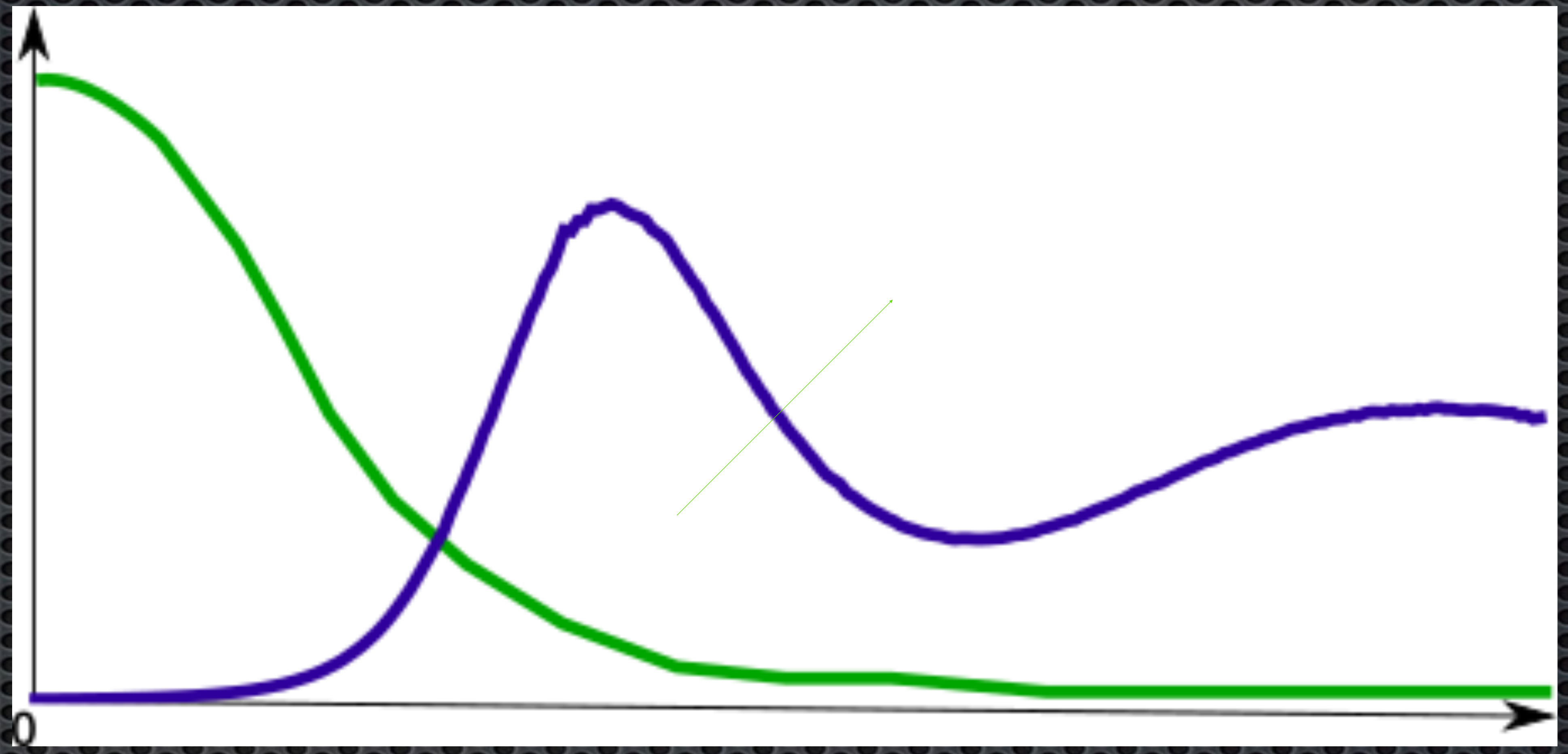


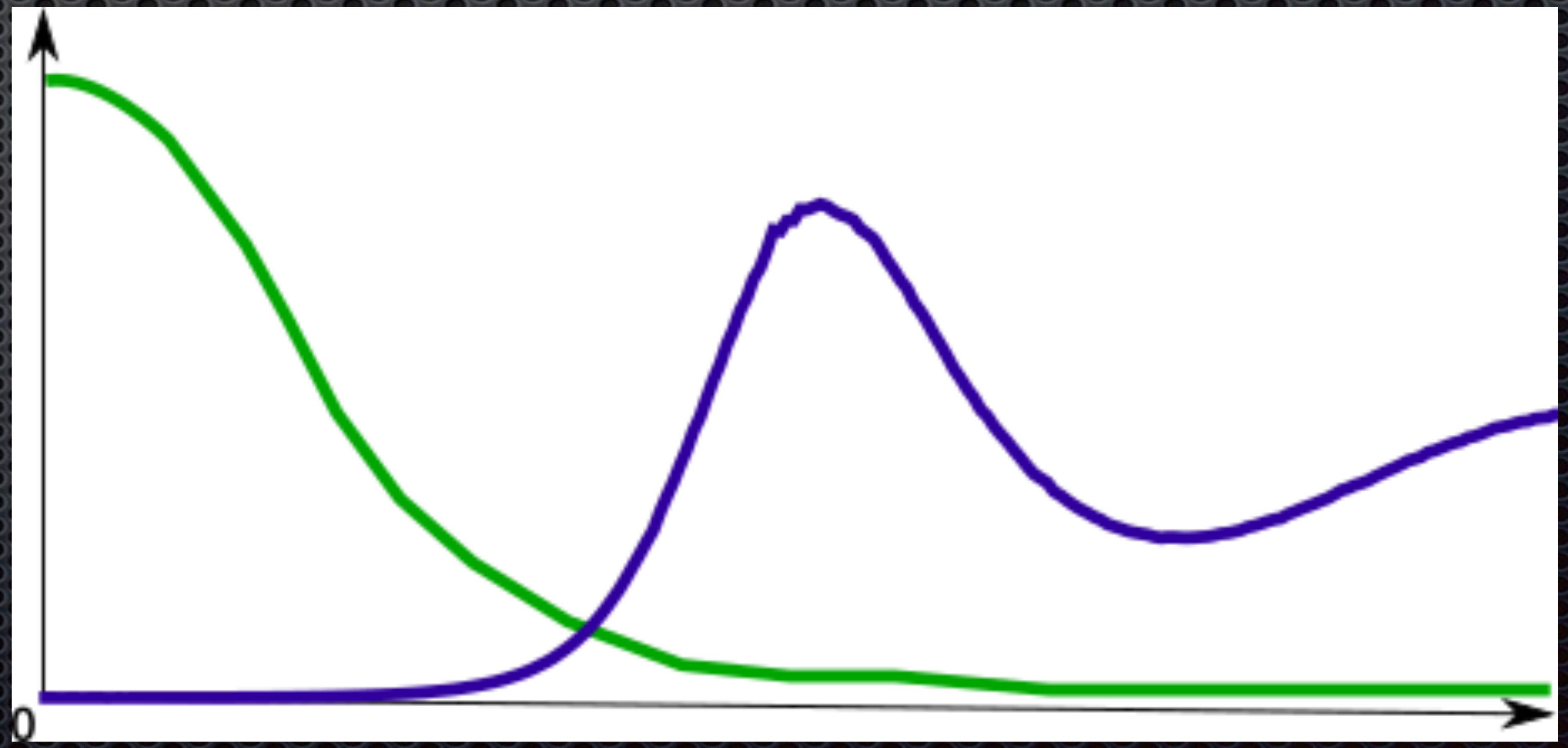
Increase in number of samples

Low Frequency zone

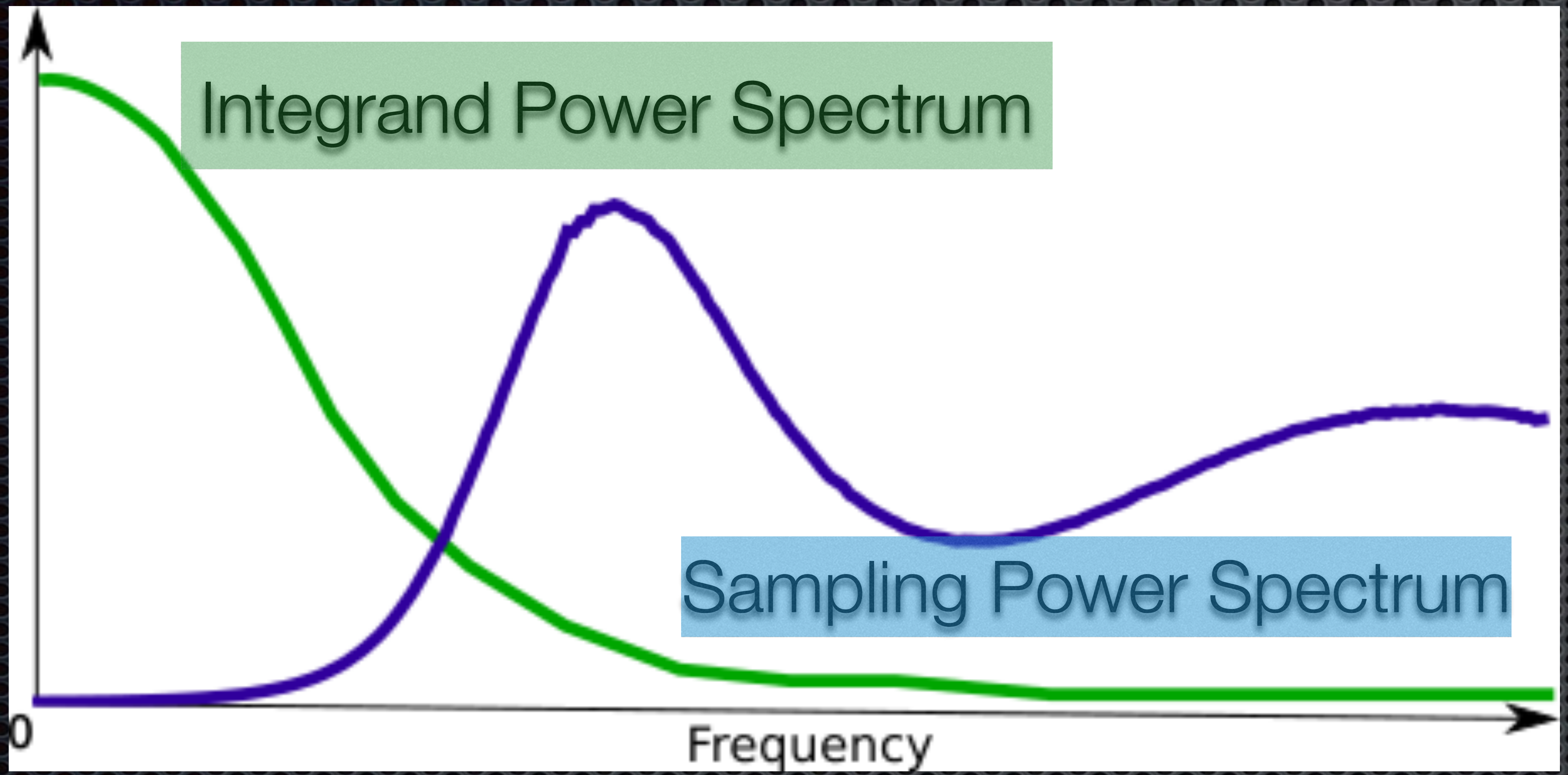


Low Frequency zone



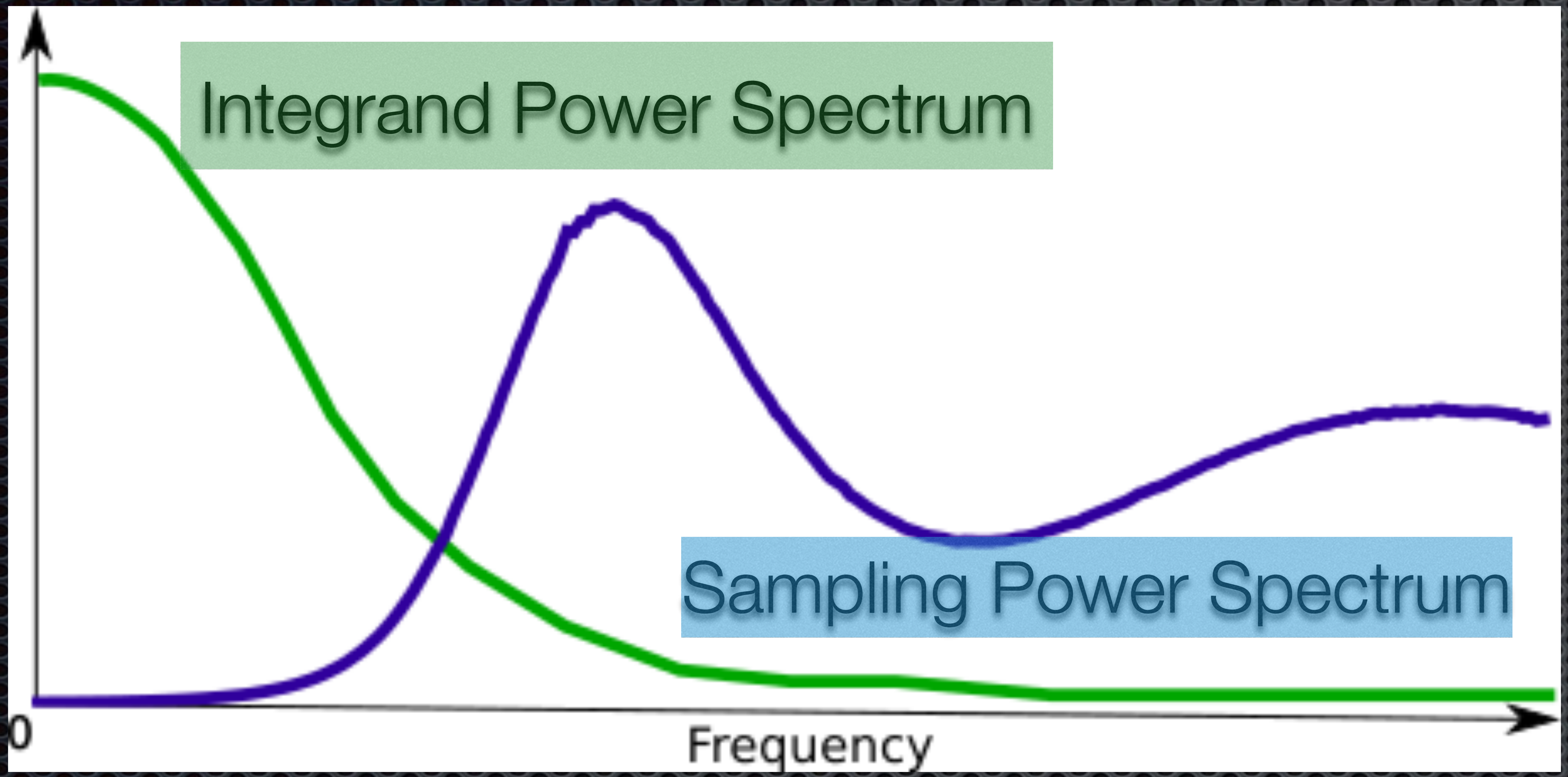


Low Frequency zone

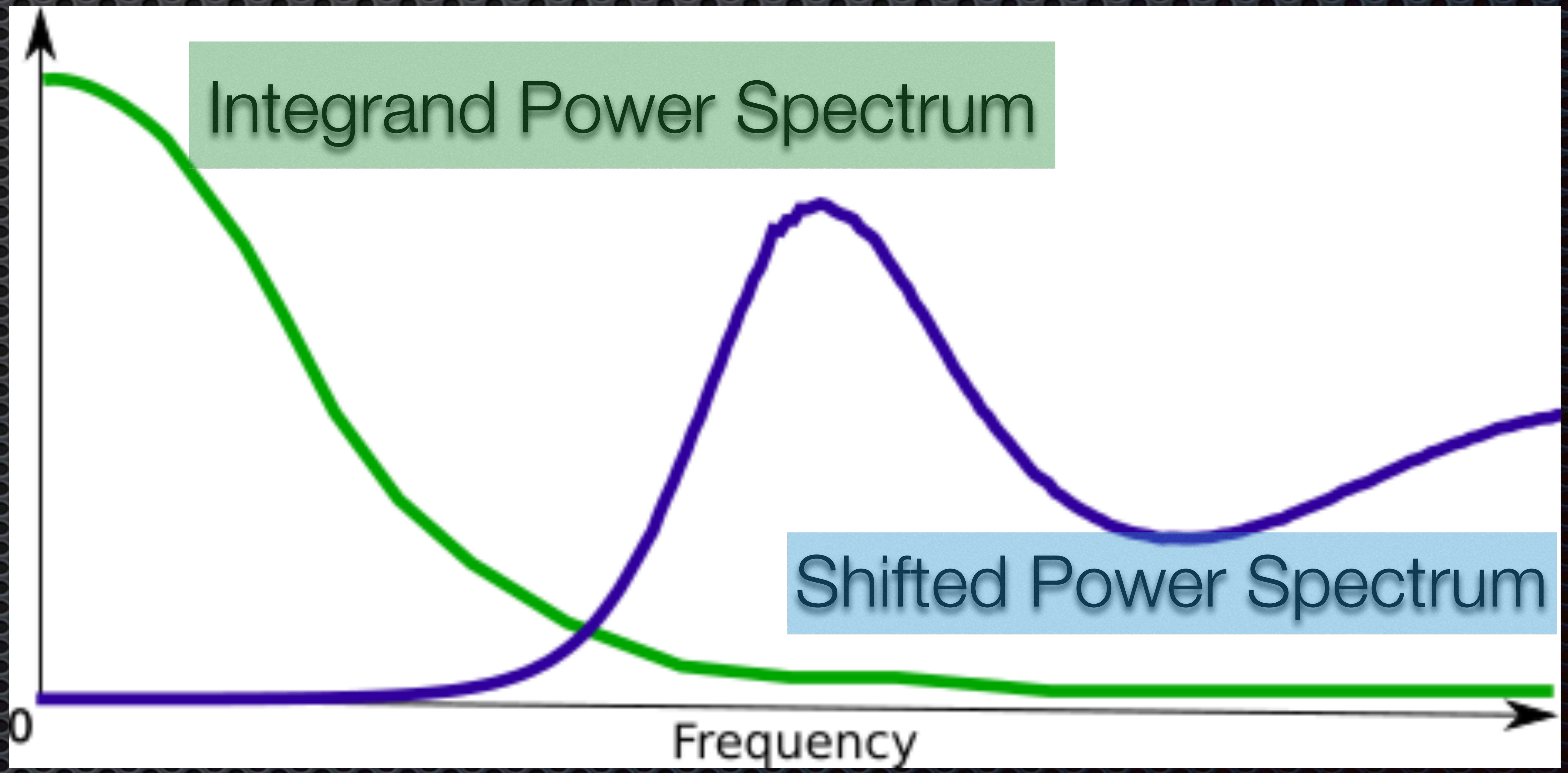


For a given number of samples

Low Frequency zone



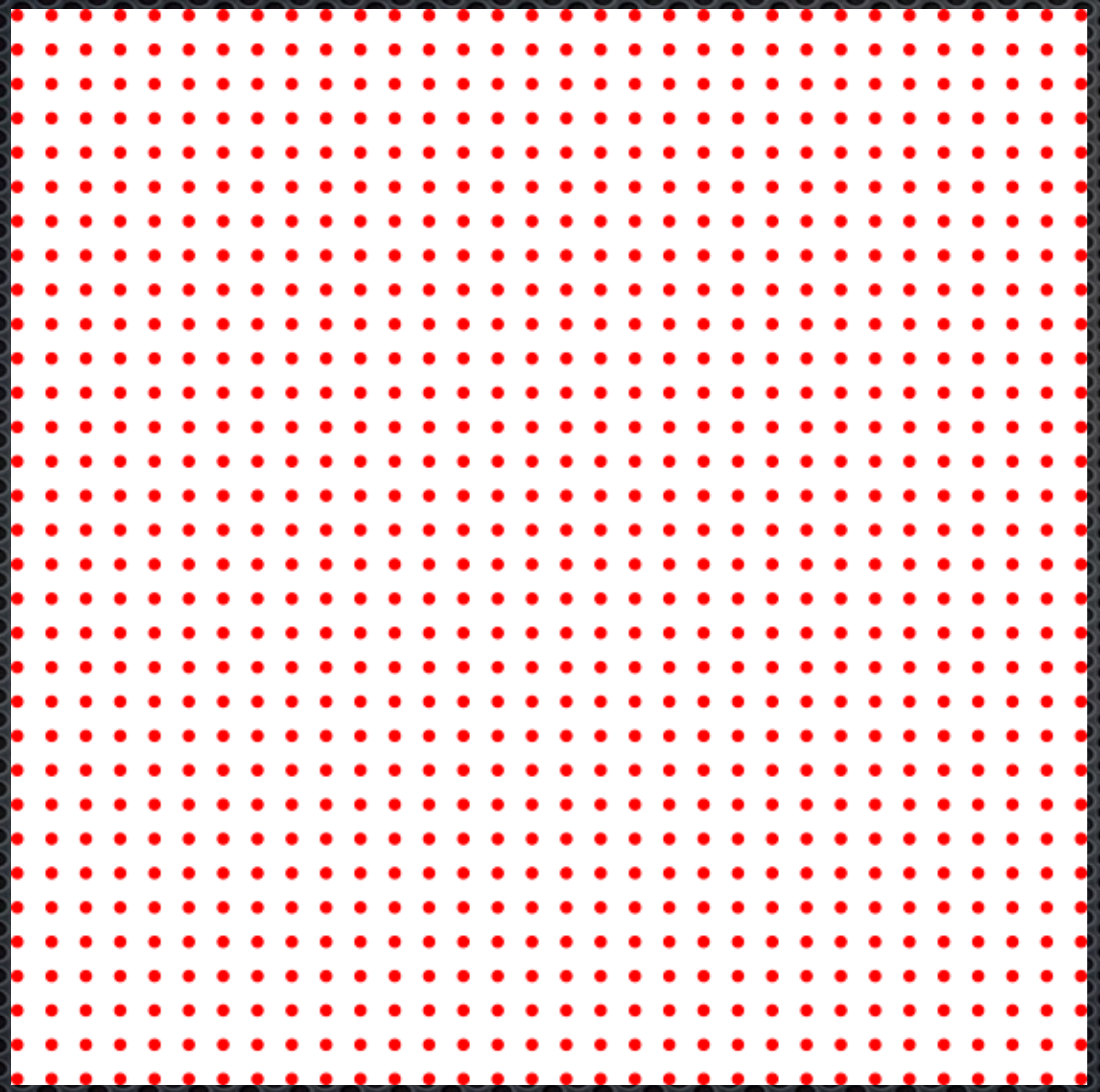
For a given number of samples



Increase in number of samples

Regular Sampling Pattern

Samples



Power Spectrum

Variance Analysis for Monte Carlo Integration

*Adrien Pilleboue[1], *Gurprit Singh[1], David Coeurjolly[2], Michael Kazhdan[3], Victor Ostromoukhov[1,2]

*Joint first Authors,
[1]Université Lyon 1, [2]CNRS/LIRIS UMR 5205, [3]Johns Hopkins University

Previous Work

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Durand [2011]

A Frequency analysis of Monte-Carlo and other numerical integration schemes

Error relates to the frequency content of samples

Previous Work

Durand [2011]

A Frequency analysis of Monte-Carlo and other numerical integration schemes

Error relates to the frequency content of samples

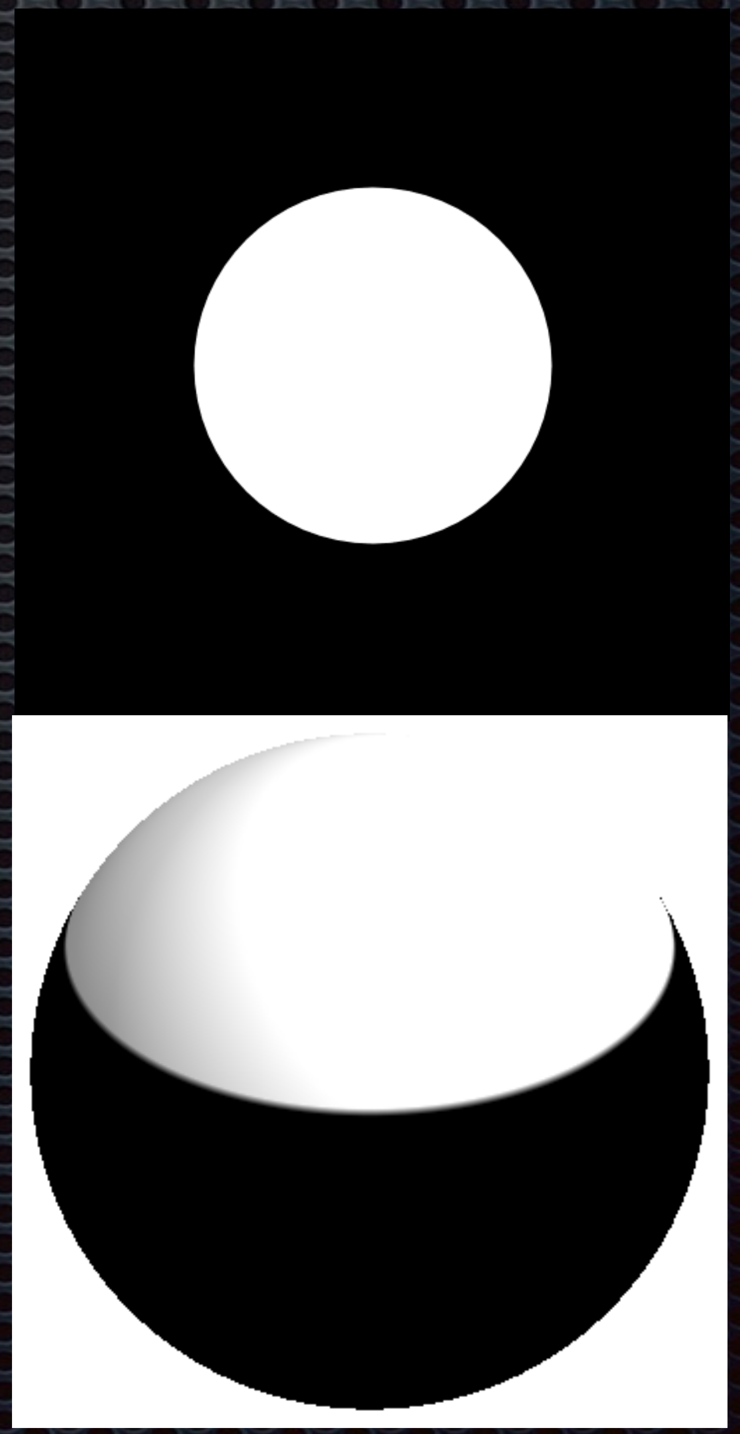
Subr and Kautz [2013]

Fourier analysis of stochastic sampling strategies for assessing bias and variance in integration

Relates variance directly to the variance of Samples' Fourier Coefficients

Integrand Power Spectrum

$\breve{\mathcal{P}}_{\mathbf{F}}(\rho)$



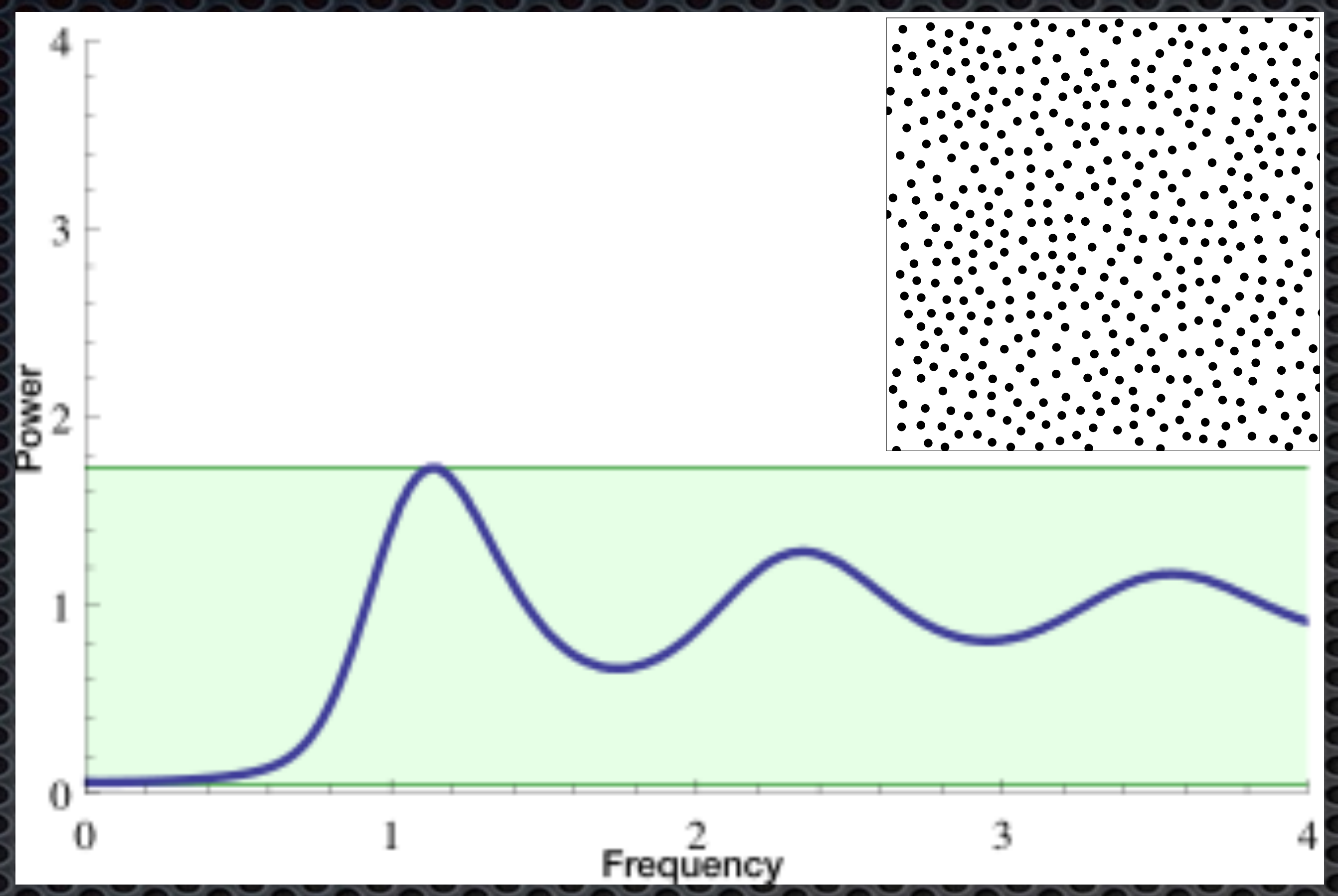
$$\breve{\mathcal{P}}_{\mathbf{F}}(\rho) = \begin{cases} c_F & \rho < \rho_0 \\ c'_F \rho^{-d-1} & \text{otherwise} \end{cases}$$

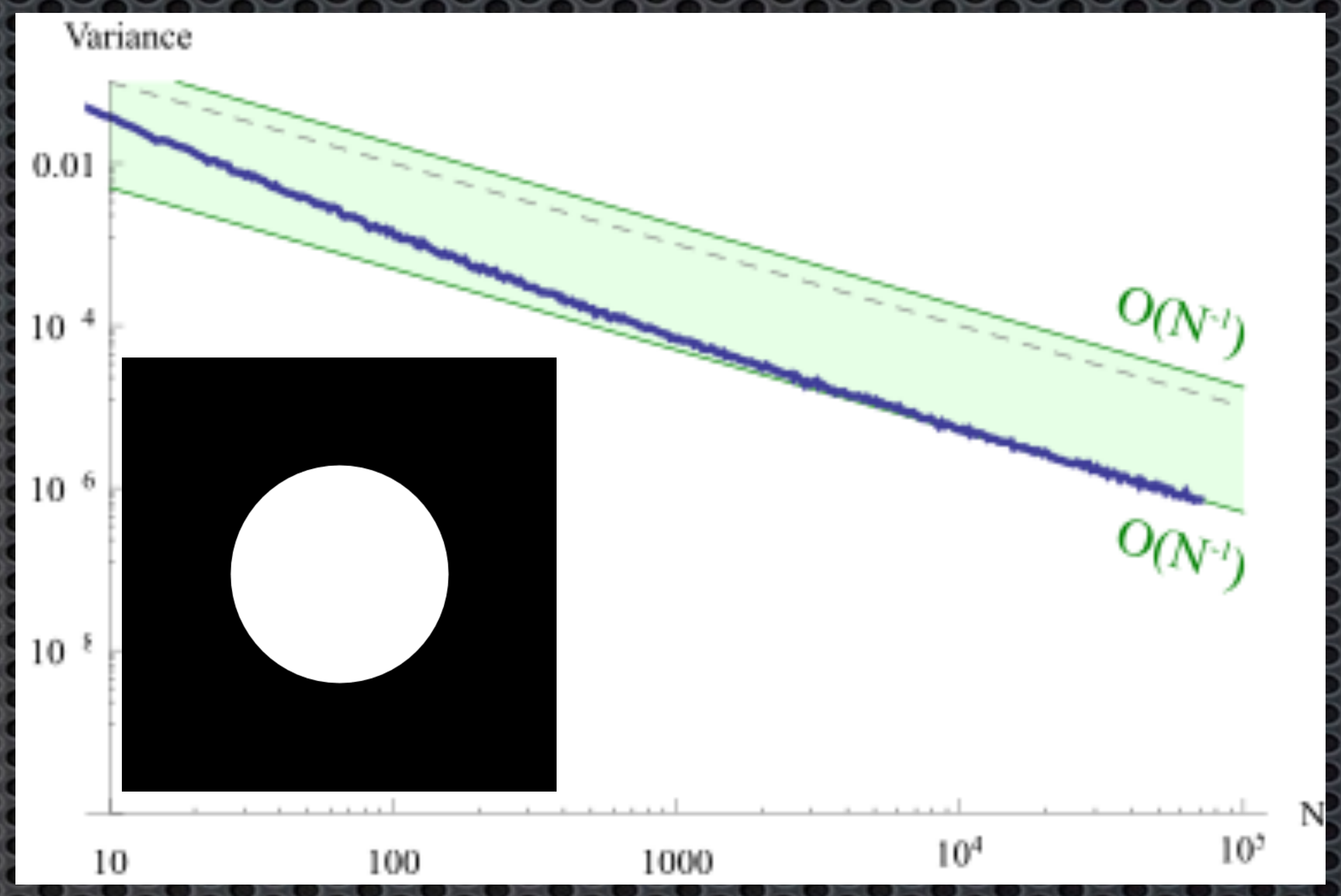
where, c_F and c'_F are constants

Brandolini et al.[2001] Mean square decay of Fourier transforms in euclidean and non euclidean spaces.

Convergence Rate Analysis

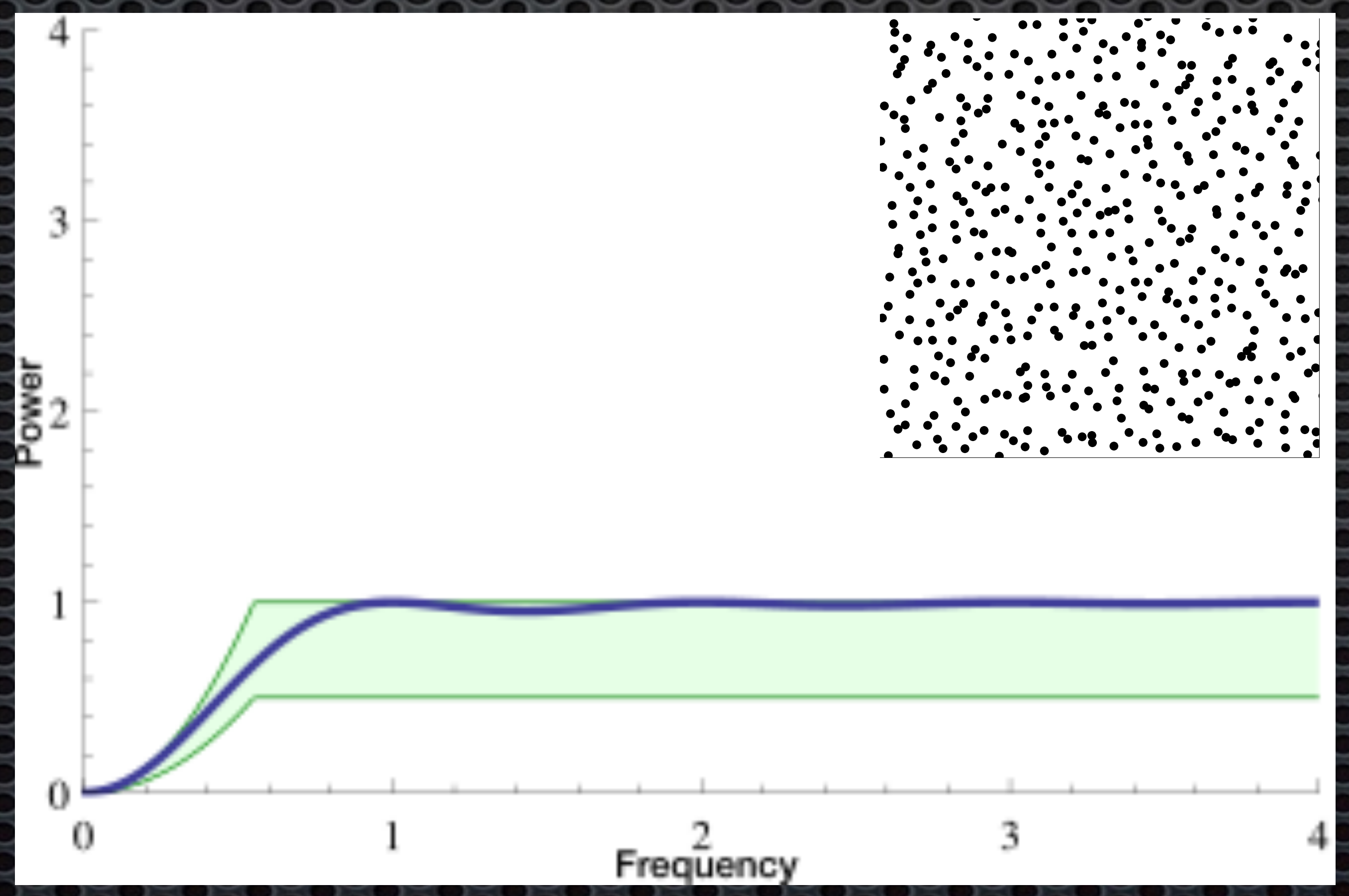
Poisson Disk

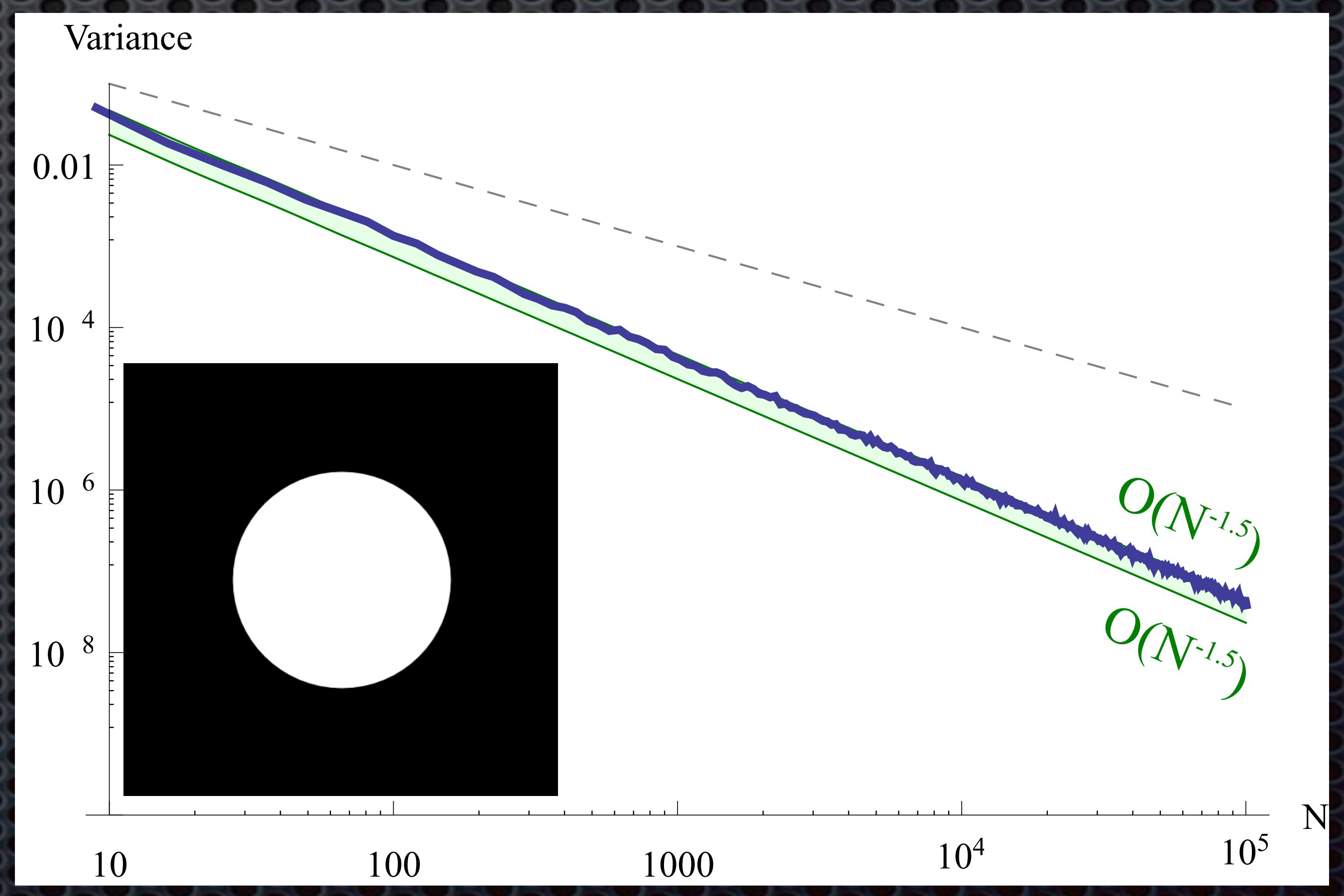




$$O\left(\frac{1}{N}\right)$$

Jittered





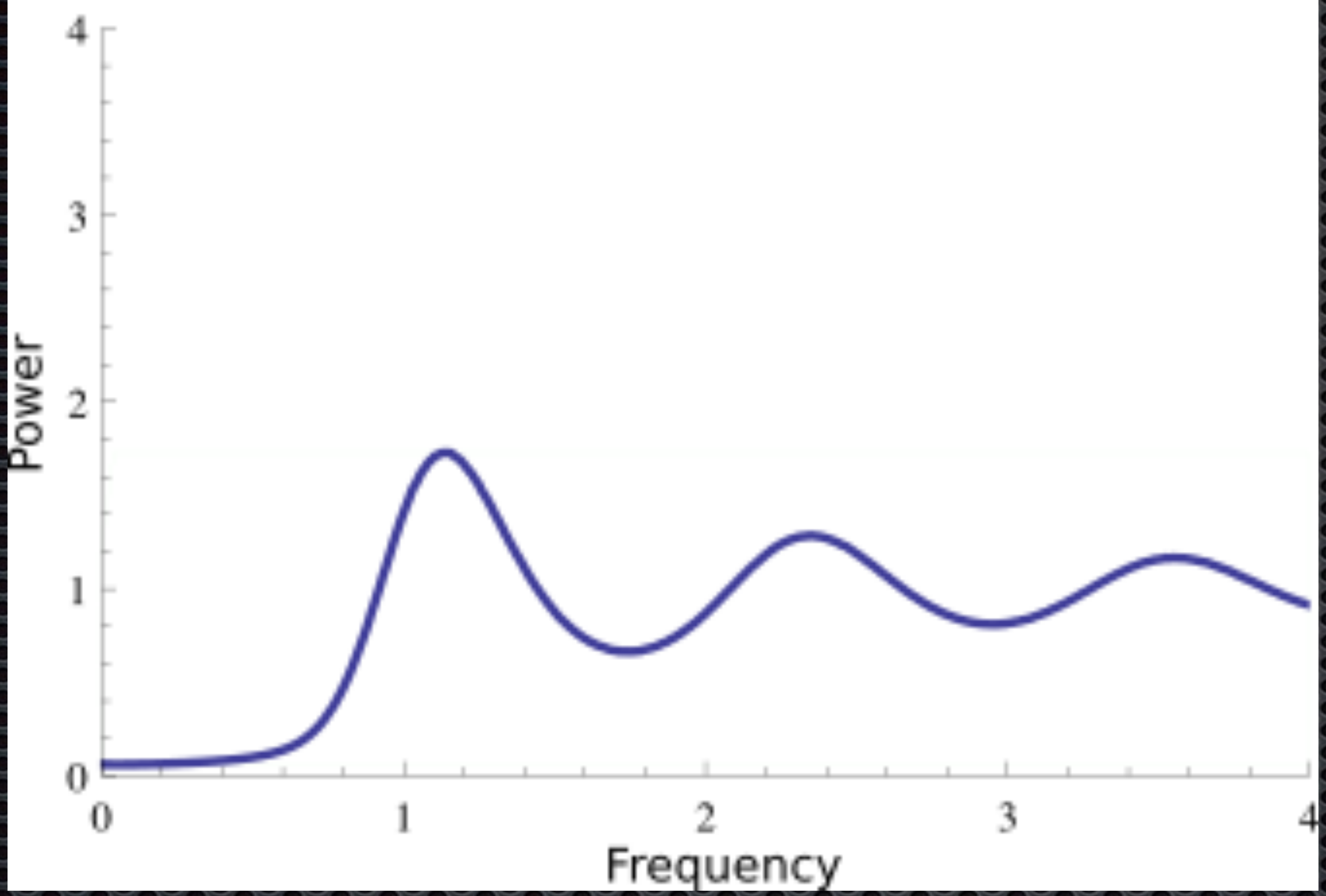
$$O\left(\frac{1}{N\sqrt{N}}\right)$$

Power Spectrum

Convergence rate

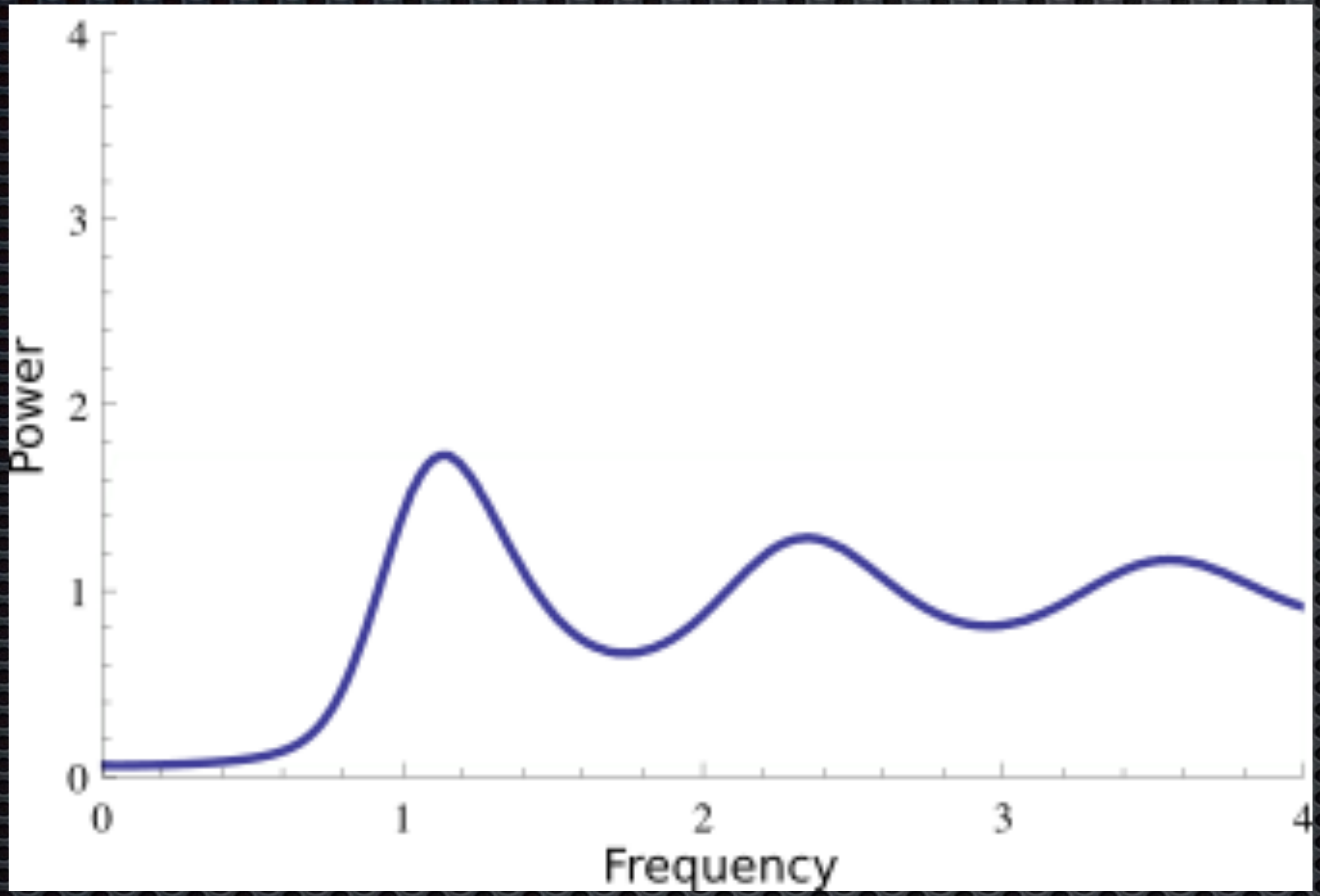
Power Spectrum Bounds

Poisson Disk

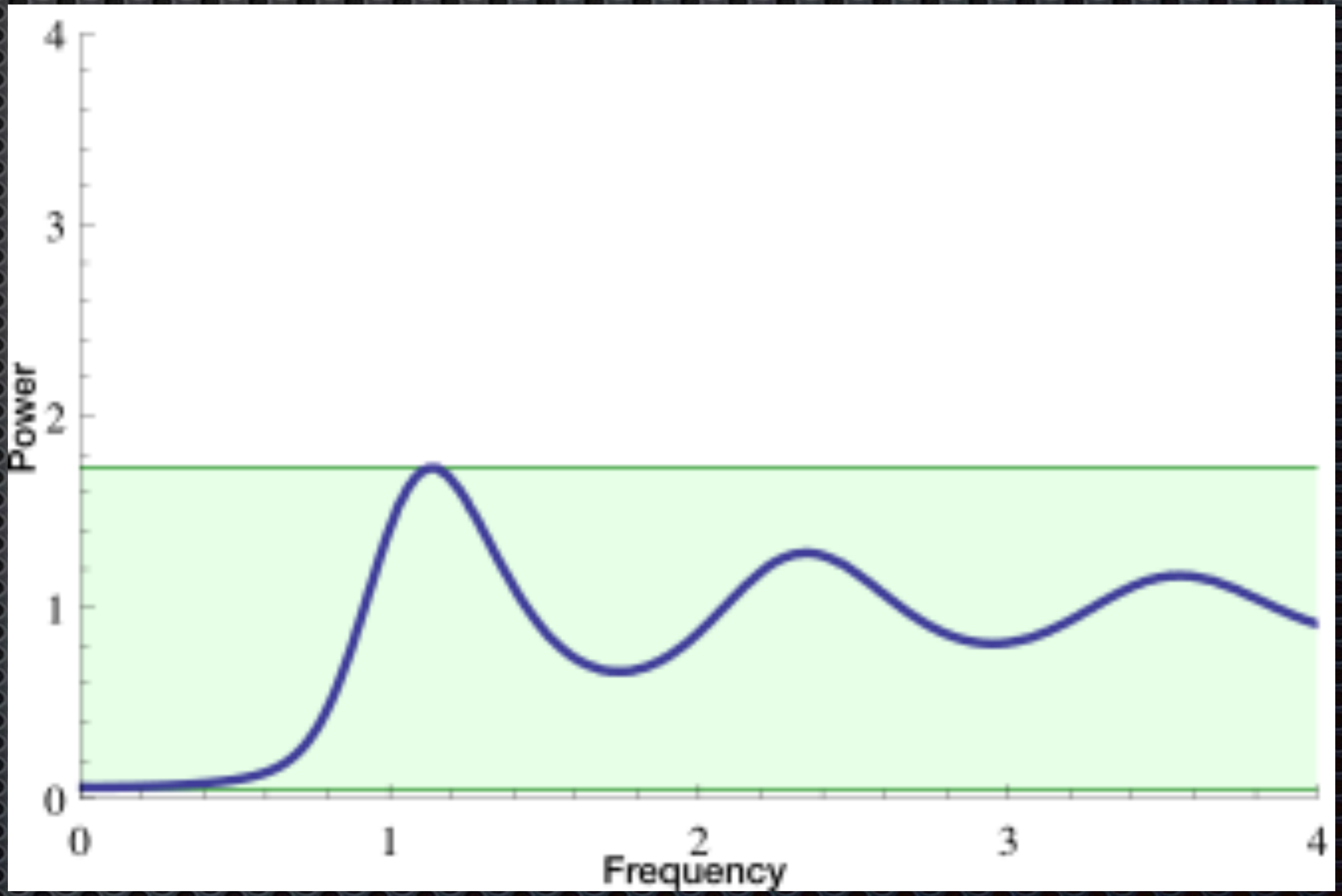


Power Spectrum Bounds

Poisson Disk



Poisson Disk



With Bounds